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Complexity and Chaos Control of a Cournot Duopoly Game with Relative Profit Maximization and Heterogeneous Expectations

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Abstract.
The paper considers a Cournot-type duopoly game, in which linear demand and cost functions are used. The two players produce differentiated goods and offer them at discrete times on a common market. In the cost functions of the players, in addition to the production cost, the cost of transporting the products is also included. Each firm does not care only about its profits but also about the percentage of its opponents' profits, using a generalized relative profit function. In this model, the players follow different strategies. More specifically, the first player is characterized as a bounded rational player while the second player follows an adaptive mechanism. The existence of the Nash Equilibrium is proved, and its stability conditions are found. The complexity that appears for some values of the game's parameters is shown. A mechanism by which the chaotic behavior of the discrete dynamical system is presented, importing a new parameter m. The algebraic results are verified, and the apparent complexity is shown by plotting bifurcation diagrams and strange attractors, computing the Lyapunov numbers, and checking the system's sensitivity on its initial conditions.

Keywords: Cournot duopoly game, discrete dynamical system, heterogeneous expectations, stability, chaotic behavior, chaos control

jel CLASSIFICATION codes
C62, C72, D43

1. INTRODUCTION

The oligopoly market is dominated by a number of companies that offer homogeneous or differentiated products. The two classic oligopoly markets are named as Cournot-model (production quantity competition) and Bertrand-model (price competition). In Cournot oligopoly models the companies try to control their production quantities in order to maximize their profits. Conversely, a company that participates in a Bertrand oligopoly market chooses its product's price and following a strategy tries to optimize

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its outputs. Officially, the first oligopoly theory was developed by Augustin Cournot in 1838, when he presented the case of an oligopoly model in which the competitors follow naïve expectations. Through these naïve expectations the players take into account their competitors' last taken values and do not consider their future reactions. The Bertrand oligopoly model is named in 1883 after Joseph Bertrand presented his game theory model by which the competitors optimize their profits by selecting prices for their products.

These first studies of Cournot and Bertrand were the first milestones for the subsequent works of many authors who contributed to the development and improvement of these models by differentiating or focusing on many of the basic assumptions of Cournot and Bertrand games. Different approaches to companies' behavior were proposed. In some duopoly models homogeneous agents were studied from some authors where they found a variety of complex dynamics, such as appearance of strange attractors [1], [4], [5], [6], [12], [19], [24], [26]. Also, heterogeneous expectations are considered in other studies [2], [3], [16], [27], [28], [29], [33], [35]. In real markets there is ignorance of the entire demand function for the producers, although it is possible that for them there is a perfect knowledge of technology, that it is represented by the cost functions. Therefore, it is more likely that firms make some local estimate of the demand. This issue has been discussed previously [11], [23], [21], [7], [8]. Efforts have been made to model bounded rationality to different economic areas: oligopoly games [2], [13], [36]; financial markets [17]; macroeconomic model such as multiplier-accelerator framework [34]. Specifically, difference equations have been used extensively to represent these economic models [15], [32]. Bounded rational players (firms) update their production strategies on discrete time periods using a local estimate of their marginal profits. With a similar local adjustment mechanism, the companies are not required to have a complete knowledge of the demand and cost functions [4], [22], [36], [8]. Also, in other studies the oligopolistic players use adaptive mechanisms that allow them to decide by a possibility as bounded rational or naïve [9]. All they need to know is whether the market responses to small production changes by an estimate of the marginal profit. All of the previous studies are mainly based on private enterprises, which seek to maximize their own profits. However, there are many companies with different ownership structures. A publicly-owned firm tends to maximize the social welfare, a semi-publicly-owned firms tend to maximize the weighted average of social welfare and also its own profit. [14], [30]. Some firms' structures are characterized by the separation between ownership and management, with managers that care about the maximization of a utility function that consists of a percentage of the opponent company's profits i.e. the generalized relative profit function [10], [31].

In this work the dynamical analysis of a Cournot-type duopoly game with differentiated products and heterogeneous expectations is presented. As it is shown the model gives complex, chaotic and unpredictable trajectories as a consequence of change in the main parameters i.e. the parameter k (players' speed of adjustment), β (second player's adaptation probability) and r (percentage domestic energy adequacy). The paper is organized by the following sections: In section 2 the main hypothesis of game are presented about the market, the competitors and their expectation strategies. After the construction of the discrete dynamical system (D.D.S) of game, the Nash Equilibrium is found and the stability conditions are found. The Nash Equilibrium coordinates and the local stability are visualized using the system's bifurcation diagrams. Strange attractors, Lyapunov number graph and sensitivity dependence on initial conditions are presented as an evidence for the chaotic behavior of D.D.S. that appears when the game's parameters take values outside the stability space. An attempt to control this chaotic orbit is made by entering a new control parameter and a dynamical analysis focusing on this parameter is presented computing and visualized the new stability conditions that allow the other parameters to take values outside their stability spaces. General conclusions concerning the behavior of the discrete dynamic system for the various values of the game parameters are presented in section 3.

2. The duopoly Game

2.1. The construction of game

This paper considers heterogeneous players more specifically, that the company 1 chooses its product's price in a rational way, following an adjustment mechanism and characterized as bounded rational player, while the company 2, by naïve way (naïve player) decides a price that maximizes its output. A classic Cournot-type duopoly market is considered, where the two companies (players) produce and offer at discrete time periods on a common market differentiated products. The players' decisions about their production are taken simultaneously at discrete time periods t = 0, 1, 2, ... At each time period t, every company form an expectation of its rival's strategy for the next time period in order to determine the corresponding profit-maximizing prices for the time period t + 1. It is supposed that q_1, q_2 are the production quantities of each company.

Also, the main hypothesis is that the consumers' preferences are represented by the following equation:

$$U\left(q_1, q_2\right) = \alpha \cdot \left(q_1 + q_2\right) - \frac{1}{2}\left(q_1^2 + q_2^2 + 2d \cdot q_1 \cdot q_2\right) (1)$$

where the market's size is expressed by positive parameter α > 0 and the differentiation degree between the two products is revealed by the parameter $d \in (-1, 1)$. For positive values of the differentiation degree (parameter d) the larger the value, the less diversification there is between their products. On the other hand, when the parameter d takes negative values it describes that both products are complementary and the minimum negative value of the differentiation parameter i.e. d = – 1, reveals that the market behave with the phenomenon of full competition. When d = 0, both products are independent and each company participates in a monopoly market. The inverse demand functions come out from the maximizing of Eq.(1) and are given as follows:

$$p_i = \alpha - q_i - d \cdot q_j \text{ with i} \neq \text{j and i, j} \boxtimes \{1{,}2\} \text{ (2)}$$

where p_i is the product price and q_i is the production quantity of firm i. In this oligopoly game linear cost functions are supposed for both players that contain the production cost and the transportation cost as follows:

The production cost is given by the following linear function:

$$C_P\left(q_i\right) = c_i \cdot q_i \text{ , with i} \boxtimes \{1{,}2\} \text{ (3)}$$

where c_i is the marginal cost for each i company and the transportation cost function is given by:

$$C_T\left(q_i\right) = \frac{f_i}{1+r} \cdot q_i + s_i \text{ , with i} \boxtimes \{1{,}2\} \text{ (4)}$$

where f_i gives the transportation price that the company i has agreed with its carrier and s_i is the fixed transport cost regardless of the quantity transported. The parameter r > – 1 expresses the percentage domestic energy adequacy (fuel) that is calculated using the following equation:

$$r = \frac{e_s - e_n}{e_n} \text{ , with } e_s > 0 \text{ (5)}$$

where e_s is the positive domestic stocked energy and e_n expresses the domestic energy needs. This transportation cost function is similar with the linear cost function that is used by Liuwei Zhao [37].

The parameter r takes positive values decreasing the transportation costs when the country has energy stock greater than its needs and negative values increasing the transportation costs when the stock is small and not enough to cover the domestic needs with the result that oil companies import fuel from abroad offering more expensive fuels on the domestic market.

With these assumptions the total cost function for each i company is given by the equation:

$$C_i\left(q_i\right) = C_P\left(q_i\right) + C_T\left(q_i\right) = c_i \cdot q_i + \tfrac{f_i}{1+r} \cdot q_i + s_i$$, with i ∈{1,2} (6)

The profit functions for two players are calculated as follows:

$$\Pi_1\left(q_1, q_2\right) = p_1 \cdot q_1 - C_1\left(q_1\right) = \left(\alpha - q_1 - d \cdot q_2\right) \cdot q_1 - c_1 \cdot q_1 - \frac{f_1}{1+r} \cdot q_1 - s_1 (7)$$

and

$$\Pi_2\left(q_1, q_2\right) = p_2 \cdot q_2 - C_2\left(q_2\right) = \left(\alpha - q_2 - d \cdot q_1\right) \cdot q_2 - c_2 \cdot q_2 - \frac{f_2}{1+r} \cdot q_2 - s_2 (8)$$

with partial derivatives:

$$\frac{\partial \Pi_1}{\partial q_1} = \alpha - c_1 - \frac{f_1}{1+r} - 2q_1 - d \cdot q_2, \frac{\partial \Pi_1}{\partial q_2} = -d \cdot q_1, \frac{\partial \Pi_2}{\partial q_2} = \alpha - c_2 - \frac{f_2}{1+r} - 2q_2 - d \cdot q_1, \frac{\partial \Pi_2}{\partial q_1} = -d \cdot q_2$$

In this study the utility function U_i is used for each i company, that is given by the following equation:

$$U_i\left(q_1, q_2\right) = (1-\mu) \cdot \Pi_i + \mu \cdot \left(\Pi_i - \Pi_j\right) = \Pi_i - \mu \cdot \Pi_j$$, with $\mu \in (0, 1)$, i $\neq$ j and i, j ∈ {1,2}

(9)

This utility function (generalized profit function) allows both players to take into account not only their individual profits, but also the opponent player's profits. An example in which a player can take into account the opponent's profits is the one who makes the decisions (manager) is a different person from the owner of the company and the first one to be a percentage (μ ∈ (0,1)) unfair and paid to make decisions which bring gains to the opponent. For the two companies it means that:

$$U_1\left(q_1, q_2\right) = \Pi_1\left(q_1, q_2\right) - \mu \cdot \Pi_2\left(q_1, q_2\right) (10)$$

and

$$U_2\left(q_1, q_2\right) = \Pi_2\left(q_1, q_2\right) - \mu \cdot \Pi_1\left(q_1, q_2\right) (11)$$

with marginal utilities the following equations:

$$\frac{\partial U_1}{\partial q_1} = \frac{\partial \Pi_1}{\partial q_1} - \mu \cdot \frac{\partial \Pi_2}{\partial q_1} = \alpha - c_1 - \frac{f_1}{1+r} - 2q_1 - d \cdot (1-\mu) \cdot q_2 (12)$$

and

$$\frac{\partial U_2}{\partial q_2} = \frac{\partial \Pi_2}{\partial q_2} - \mu \cdot \frac{\partial \Pi_1}{\partial q_2} = \alpha - c_2 - \frac{f_2}{1+r} - 2q_2 - d \cdot (1-\mu) \cdot q_1 (13)$$

As it is noticed the two players follow different strategies to choose the production quantitiesq_i. The first player is supposed as bounded rational player deciding to increase the level of adaptation in the mechanism if there is a positive marginal utility for him, or to decrease the level of adaptation if his marginal utility is negative. According to the existing literature this mechanism described by the following dynamical equation:

$$\frac{q_1(t+1) - q_1(t)}{q_1(t)} = k \cdot \frac{\partial U_1}{\partial q_1} (14)$$

where the positive parameter k > 0, expresses the speed of adjustment of first company and it gives the extent q_1 quantity variation of the company following a given utility signal. Moreover it captures the fact that relative variations of the quantity are proportional to the marginal utility. On the other hand the second player is characterized as an adaptive player who decides with probability β to behave as a bounded rational player and with probability 1-β as a naïve player. The second player's expectations are described by the dynamical equation:

$$\begin{aligned} q_2(t+1) &= \beta \cdot \left(q_2(t) + k \cdot q_2(t) \cdot \tfrac{\partial U_2}{\partial q_2} \right) \\ &\quad + (1-\beta) \cdot \tfrac{1}{2} \cdot \left(\alpha - c_2 - \tfrac{f_2}{1+r} - 2q_2 - d \cdot (1-\mu) \cdot q_1(t) \right) \end{aligned} \quad (15)$$

The final stage of the game's construction is to find the discrete dynamical system that contains the requirement of the players' expectations. The discrete dynamical system of the duopoly game if given by:

$$\begin{cases} q_1(t+1) &= q_1(t) + k \cdot q_1(t) \cdot \frac{\partial U_1}{\partial q_1} \\ q_2(t+1) &= \beta \cdot \left(q_2(t) + k \cdot q_2(t) \cdot \frac{\partial U_2}{\partial q_2} \right) \\ &\quad + (1-\beta) \cdot \frac{1}{2} \cdot \left(\alpha - c_2 - \frac{f_2}{1+r} - 2q_2(t) - d \cdot (1-\mu) \cdot q_1(t) \right) \end{cases} \quad (16)$$

This work focuses on the dynamical analysis of this discrete dynamical system of Eq.(16) with respect to the parameters k, β and r.

2.2. The Nash Equilibrium

The Nash Equilibrium position can be calculated using the static game's algebraic system that is described by the following equation:

$$\begin{cases} \frac{\partial U_1}{\partial q_1} = 0 \\ \frac{\partial U_2}{\partial q_2} = 0 \end{cases} (17)$$

The unique solution of this system is the position (q_1, q_2) where:

$$q_1 = \frac{2(\alpha - c_1) - 2\cdot\frac{f_1}{1+r} - d\cdot(1-\mu)\cdot\left(\alpha - c_2 - \frac{f_2}{1+r}\right)}{4 - d^2\cdot(1-\mu)^2},$$
$$q_2 = \frac{2(\alpha - c_2) - 2\cdot\frac{f_2}{1+r} - d\cdot(1-\mu)\cdot\left(\alpha - c_1 - \frac{f_1}{1+r}\right)}{4 - d^2\cdot(1-\mu)^2}$$

After the replacement of these values in the two equations of the dynamical game Eq.(16) , it is verified that they are also solutions of the discrete dynamical system and the position $E_*\left(q_1^*, q_2^*\right)$ can characterized as the Nash equilibrium position of the dynamical game's system where:

$$q_i^* = \frac{2(\alpha - c_i) - 2\cdot\frac{f_i}{1+r} - d\cdot(1-\mu)\cdot\left(\alpha - c_j - \frac{f_j}{1+r}\right)}{4 - d^2\cdot(1-\mu)^2}$$, with $i \neq j$ and i, j {1, 2} (18)

2.3. Local stability conditions of Nash Equilibrium

The study of local stability of the E_* Nash Equilibrium position of Eq.(16) needs the calculation of the Jacobian matrix. The Jacobian matrix of the discrete dynamical system of Eq.(16) is given by following table:

$$J = \begin{bmatrix} 1 + k\cdot\left(q_1\cdot\frac{\partial^2 U_1}{\partial q_1^2} + \frac{\partial U_1}{\partial q_1}\right) & k\cdot q_1\cdot\frac{\partial^2 U_1}{\partial q_1 \partial q_2} \\ k\cdot\beta\cdot q_2\cdot\frac{\partial^2 U_2}{\partial q_2 \partial q_1} - \frac{1}{2}\cdot d\cdot(1-\mu)\cdot(1-\beta) & \beta + k\cdot\beta\cdot\left(q_2\cdot\frac{\partial^2 U_2}{\partial q_2^2} + \frac{\partial U_2}{\partial q_2}\right) \end{bmatrix} (19)$$

and the Jacobian matrix of the Nash Equilibrium position E_* is described as follows:

$$J\left(E_*\right) = \begin{bmatrix} 1 + k\cdot q_1^*\cdot\frac{\partial^2 U_1}{\partial q_1^2} & k\cdot q_1^*\cdot\frac{\partial^2 U_1}{\partial q_1 \partial q_2} \\ k\cdot\beta\cdot q_2^*\cdot\frac{\partial^2 U_2}{\partial q_2 \partial q_1} - \frac{1}{2}\cdot d\cdot(1-\mu)\cdot(1-\beta) & \beta + k\cdot\beta\cdot\left(q_2^*\cdot\frac{\partial^2 U_2}{\partial q_2^2} + \frac{\partial U_2}{\partial q_2}\right) \end{bmatrix} (20)$$

with

$$Tr(J) = 1 + \beta - 2k\cdot q_1^* - 2k\cdot\beta\cdot q_2^* (21)$$

and

$$Det(J) = \beta - 2k\cdot\beta\cdot q_1^* - \frac{1}{2}k\cdot(1-\beta)\cdot d^2\cdot(1-\mu)^2\cdot q_1^* - 2k\cdot\beta\cdot q_2^* + k^2\cdot\beta\left[4 - d^2\cdot(1-\mu)^2\right]\cdot q_1^*\cdot q_2^* (22)$$

The Nash Equilibrium position is locally asymptotically stable if the following conditions are hold simultaneously [18], [15], [20]:

$$(i):\quad 1 - Det(J) > 0$$
$$(ii):\quad 1 - Tr(J) + Det(J) > 0 \quad (23)$$
$$(iii):\quad 1 + Tr(J) + Det(J) > 0$$

The inequality (i) gives:

$$1 - Det(J) > 0 \Leftrightarrow$$

$$\Leftrightarrow 1-\beta+2k\cdot\beta\cdot q_1^*+\frac{1}{2}k\cdot(1-\beta)\cdot d^2\cdot(1-\mu)^2\cdot q_1^*+2k\cdot\beta\cdot q_2^*+k^2\cdot\beta\cdot\left[4-d^2\cdot(1-\mu)^2\right]\cdot q_1^*\cdot q_2^* > 0(24)$$

That is the 1^{st} local stability condition.

It's easy to prove that the second inequality (ii) is always satisfied because:

$$1 - Tr(J) + Det(J) > 0 \Leftrightarrow k\cdot q_1^*\left(2k\cdot\beta\cdot q_2^* + 1 - \beta\right)\cdot\left[4 - d^2(1-\mu)^2\right] > 0$$

The third inequality (iii) gives:

$$1 + Tr(J) + Det(J) > 0 \Leftrightarrow$$

$$\Leftrightarrow 2k^2\cdot\beta\cdot\left[4-d^2\cdot(1-\mu)^2\right]\cdot q_1^*\cdot q_2^* - k\cdot q_1^*\left[4+4\beta+d^2\cdot(1-\mu)^2\cdot(1-\beta)\right]$$
$$-8k\cdot\beta\cdot q_2^* + 4 + 4\beta > 0(25)$$

That is the 2^{nd} local stability condition of the Nash Equilibrium position.

Proposition 1:

The Nash equilibrium position $E_* = \left(q_1^*, q_2^*\right)$ *of the discrete dynamical system of Eq.(16) is locally asymptotically stable if:*

$$1-\beta+2k\cdot\beta\cdot q_1^*+\frac{1}{2}k\cdot(1-\beta)\cdot d^2\cdot(1-\mu)^2\cdot q_1^*+2k\cdot\beta\cdot q_2^*+k^2\cdot\beta\cdot\left[4-d^2\cdot(1-\mu)^2\right]\cdot q_1^*\cdot q_2^* > 0$$

and

$$2k^2\cdot\beta\cdot\left[4-d^2\cdot(1-\mu)^2\right]\cdot q_1^*\cdot q_2^*-k\cdot q_1^*\left[4+4\beta+d^2\cdot(1-\mu)^2\cdot(1-\beta)\right]-8k\cdot\beta\cdot q_2^*+4+4\beta > 0$$

where

$$q_1^* = \frac{2\left(\alpha - c_1\right) - 2\cdot\frac{f_1}{1+r} - d\cdot(1-\mu)\cdot\left(\alpha - c_2 - \frac{f_2}{1+r}\right)}{4 - d^2\cdot(1-\mu)^2}$$

and

$$q_2^* = \frac{2\left(\alpha - c_2\right) - 2\cdot\frac{f_2}{1+r} - d\cdot(1-\mu)\cdot\left(\alpha - c_1 - \frac{f_1}{1+r}\right)}{4 - d^2\cdot(1-\mu)^2}$$

2.4. Numerical simulations

In this subsection some numerical simulations focusing on the parameters k, β and r are presented to verify the previous algebraic results about the local stability conditions

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of the Nash Equilibrium position and to reveal the system's chaotic behavior that is appeared when the game's parameters take values outside their stability spaces. At first 2D and 3D stability regions between these parameters are presented. With respect to each one of these parameters the bifurcations diagrams, strange attractors and sensitive dependence on initial conditions are plotted.

2.4.1. Numerical simulations focusing on the parameter k

The Figure 16 shows the 3D local stability region of the three main game's parameters, the parameter k (players' speed of adjustment), the parameter β (second player's adaptation probability) and the parameter r (percentage domestic energy adequacy) setting specific values to the other parameters as follows: $\alpha = 5$, $c_1 = 1$, $c_2 = 1.3$, $f_1 = 0.5$, $f_2 = 0.7$, $d = 0.5$ and $\mu = 0.5$.

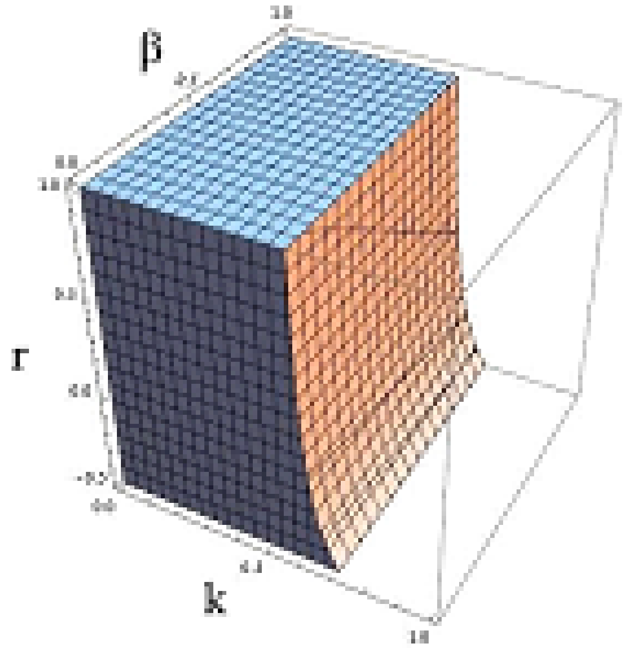

Figure 1: 3D Stability region of the Nash Equilibrium of Eq.(16) between the parameters k, r and β.

To make the numerical simulations focusing on the parameter k, specific values for the other parameters are choosing. For example setting the values of the parameters: $\alpha = 5$, $c_1 = 1$, $c_2 = 1.3$, $f_1 = 0.5$, $f_2 = 0.7$, $d = 0.5$, $\mu = 0.5$ and $r = 0.1$, it gives the following Nash Equilibrium position:

$$E_* = \left(q_1^*, q_2^*\right) \simeq (1.61\ ,\ 1.33)\,(26)$$

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The stability conditions focusing on the parameters k and β become as:

$$8.43 \cdot k^2 \cdot \beta - 5.83 \cdot k \cdot \beta - 0.05 \cdot k + \beta - 1 < 0(27)$$

and

$$16.86 \cdot k^2 \cdot \beta - 16.98 \cdot k \cdot \beta - 6.54 \cdot k + 4 \cdot \beta + 4 > 0(28)$$

In Figure 2 the region of common solutions of Eq.(27) and Eq.(28) is plotted. A useful result for the economists is that a space of local stability is created where the Nash Equilibrium E_* is locally asymptotically stable for every value of the parameters β while the values of the parameter k belong to a close interval.

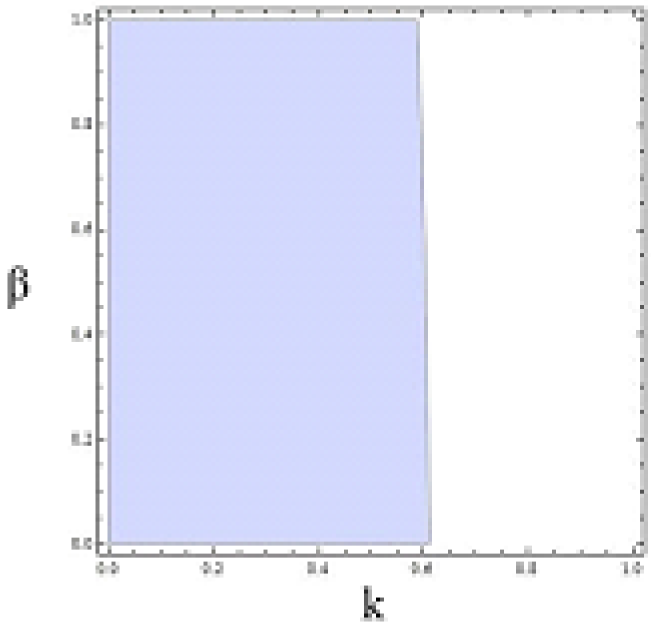

Figure 2: Stability region of the Nash Equilibrium of Eq.(16) for α = 5, c_1 = 1, c_2 = 1.3, f_1 = 0.5, f_2 = 0.7, d = 0.5, μ = 0.5 and r = 0.1.

Choosing a specific value for the parameter β = 0.3, the local stability conditions becomes as follows:

k ∈ (0,0.99) (1^{st} stability condition) and k ∈ (0,0.6) (2^{nd} stability condition)

with common solutions:

k ∈ (0,0.6) (29)

that is the final local stability condition of the Nash Equilibrium position for these parameters' values.

This algebraic result is verified by stability region between the parameter k (horizontal axis) and the parameter β (vertical axis) that is plotted in Figure 2. Also, the

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bifurcation diagrams (Fig.3 and Fig.4) show that there is locally asymptotically stable Nash Equilibrium position with coordinates $q_1^* \simeq 1.61$ and $q_2^* \simeq 1.33$ for values of the parameter k lower than 0.60 and after this value the position E_* becomes unstable through period-doubling bifurcation diagrams and for larger values of the parameter k, complex dynamics behavior is observed such as cycles of higher order and chaos.

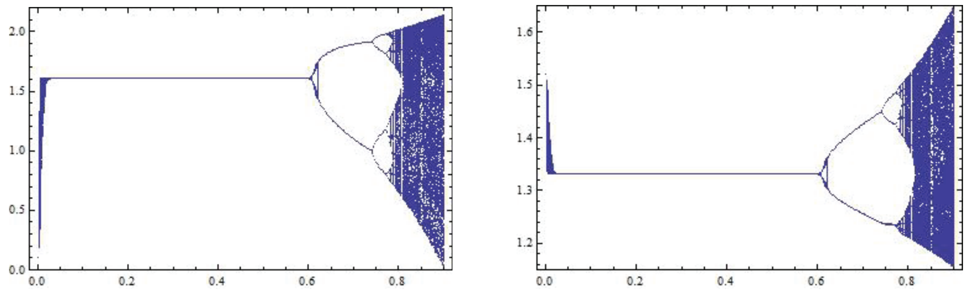

Figure 3: Bifurcation diagrams with respect to the parameter k against the variables q_1^* (left) and q_2^* (right) with 400 iterations of the map Eq.(16) for $\alpha = 5$, $c_1 = 1$, $c_2 = 1.3$, $f_1 = 0.5$, $f_2 = 0.7$, $\beta = 0.3$, $d = 0.5$, $\mu = 0.5$ and $r = 0.1$.

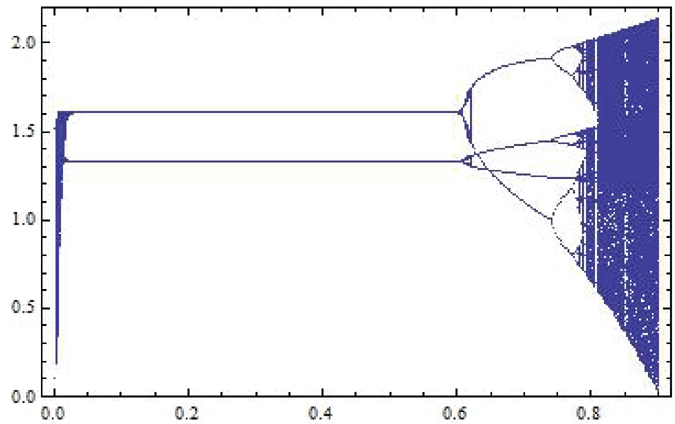

Figure 4: The two bifurcation diagrams of Fig.3 are plotted in one.

Figure 5 shows the graphs of the same orbit (strange attractors) and Lyapunov numbers' diagram of the orbit of (0.1,0.1) for $\alpha = 5$, $c_1 = 1$, $c_2 = 1.3$, $f_1 = 0.5$, $f_2 = 0.7$, $\beta = 0.3$, $d = 0.5$, $\mu = 0.5$, $r = 0.1$ and $k = 0.9$. These results show that when all parameters are fixed and only k is varied the structure of the game becomes complicated through period doubling bifurcations, more complex bounded attractors are created which are aperiodic cycles of higher order or chaotic attractors.

To demonstrate the sensitivity on initial conditions of the system Eq.(16), two orbits with initial points (0.1,0.1) and (0.101,0.1), respectively are computed. Figure 6 shows sensitive dependence on initial conditions for x-coordinate of the two orbits, for the system Eq.(16), plotted against the time with the parameter values. As in first case

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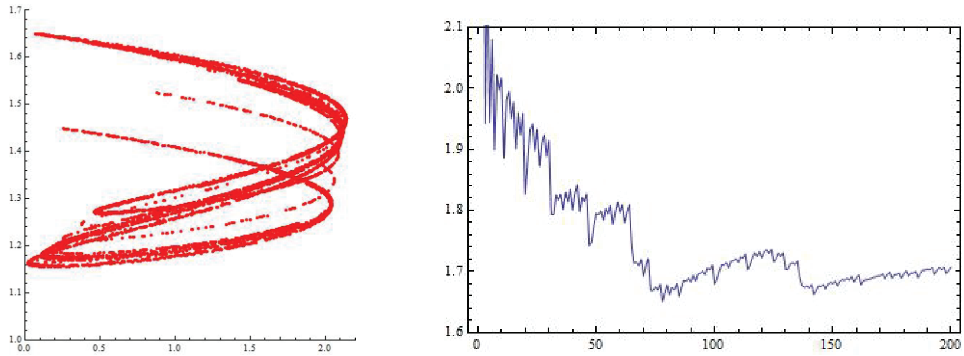

Figure 5: Phase portrait (strange attractor) (left) and Lyapunov numbers (right) of the orbit of (0.1,0.1) with 8000 iterations of the map Eq.(16) for α = 5, c_1 = 1, c_2 = 1.3, f_1 = 0.5, f_2 = 0.7, β = 0.3, d = 0.5, μ = 0.5, r = 0.1 and k = 0.9.

also, here at the beginning the time series are indistinguishable; but after a number of iterations, the difference between them builds up rapidly.

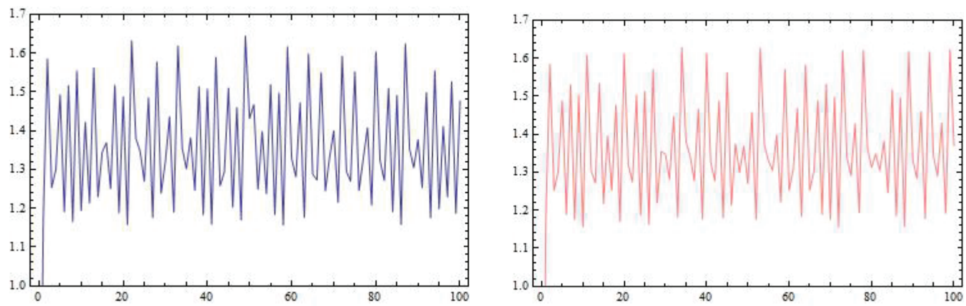

Figure 6: Sensitive dependence on initial conditions for q_1 -coordinate plotted against the time: the orbit of (0.1,0.1) (left) and the orbit of (0.101,0.1) (right) of the system Eq.(16) for α = 5, c_1 = 1, c_2 = 1.3, f_1 = 0.5, f_2 = 0.7, β = 0.3, d = 0.5, μ = 0.5, r = 0.1 and k = 0.9.

2.4.2. Numerical simulations focusing on the parameter □

Setting specific values to the following parameters: α = 5, c_1 = 1, c_2 = 1.3, f_1 = 0.5, f_2 = 0.7, d = 0.5, μ = 0.5, k = 0.75 and r = 0.1, the bifurcation diagrams against the variables q_1 (Fig.7-left) and q_2 (Fig.8-right) with respect to the parameter β are plotted. The larger the values of the parameter β, period-doubling bifurcations are appeared.

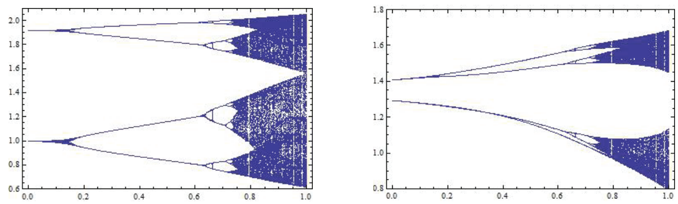

Figure 7: Bifurcation diagrams with respect to the parameter β against the variables q_1^* (left) and q_2^* (right) with 400 iterations of the map Eq.(16) for α = 5, c_1 = 1, c_2 = 1.3, f_1 = 0.5, f_2 = 0.7, d = 0.5, μ = 0.5, k = 0.75 and r = 0.1.

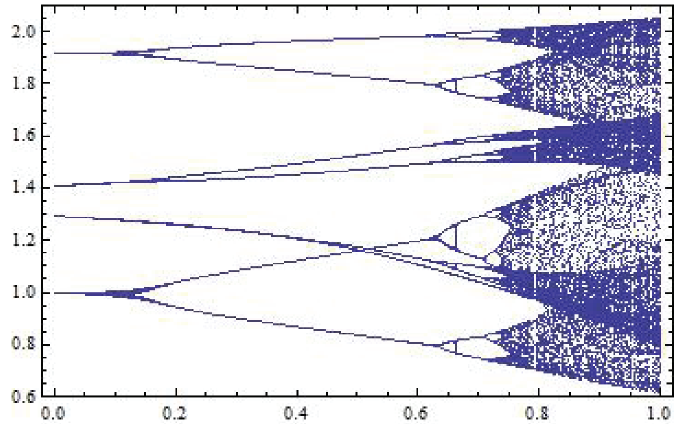

Figure 8: The two bifurcation diagrams of Fig.7 are plotted in one.

For large values of the parameter β, chaotic behavior of the system is caused. As an evidence of this chaotic behavior of the dynamical system strange attractors (Fig.9-left) and Lyapunov numbers (Fi. 9-right)greater than 1.00 are presented for a large value of the parameter $\beta = 0.99$.

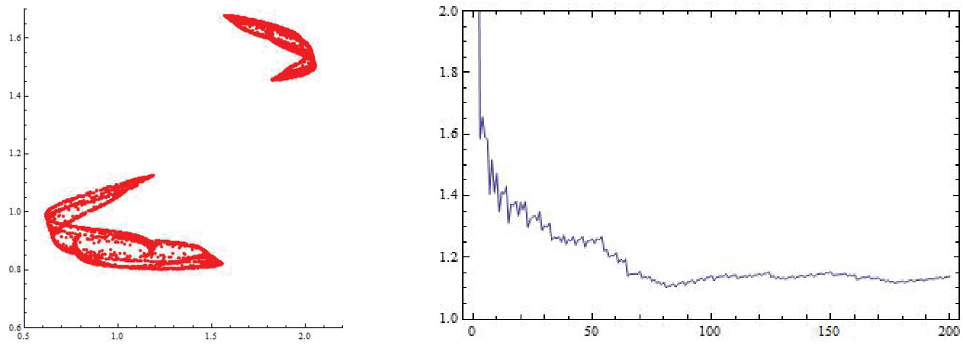

Figure 9: Phase portrait (strange attractor) (left) and Lyapunov numbers (right) of the orbit of (0.1,0.1) with 8000 iterations of the map Eq.(16) for $\alpha = 5$, $c_1 = 1$, $c_2 = 1.3$, $f_1 = 0.5$, $f_2 = 0.7$, $d = 0.5$, $\mu = 0.5$, $k = 0.75$, $r = 0.1$ and $\beta = 0.99$.

A sensitivity analysis of the dynamical system of Eq.(16) on initial conditions for the same value of the parameter $\beta = 0.99$ (outside the stability space) is presented in Figures 10. Two different initial conditions are supposed with a small difference at the first q_1-coordinate. The time series of the system with initial condition (0.1,0.1) (Fig.10-left) and initial condition (0.101,0.1) (Fig.10-right) have the same route at the first iterations. After a number of iterations the differences between them are appeared showing that the dynamical system becomes sensitive on initial conditions when the parameter β takes values outside the stability space.

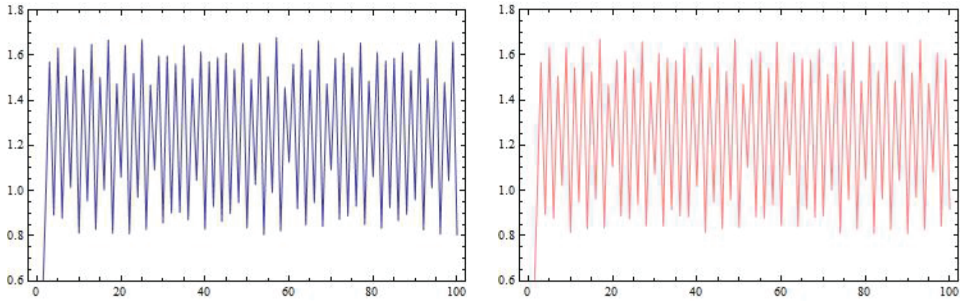

Figure 10: Sensitive dependence on initial conditions for q_1 -coordinate plotted against the time: the orbit of (0.1,0.1) (left) and the orbit of (0.101,0.1) (right) of the system Eq.(16) for α = 5, c_1 = 1, c_2 = 1.3, f_1 = 0.5, f_2 = 0.7, d = 0.5, μ = 0.5, k = 0.75, r = 0.1 and β = 0.99.

2.4.3. Numerical simulations focusing on the parameter r

Numerical simulations are presented with respect to the parameter r (percentage domestic energy adequacy). Specific values for all other parameters are set as follows: α = 5, c_1 = 1, c_2 = 1.3, f_1 = 0.5, f_2 = 0.7, β = 0.3, d = 0.5, μ = 0.5 and k = 0.76. In Figure 11 the bifurcation diagrams against the variables of q_1 (left) and q_2 (left) are plotted with respect to the parameter r. As the parameter k takes values until a specific -0.60 there is a locally asymptotically stable Nash Equilibrium and after this value period-doubling bifurcations are appeared and for values of the parameter k larger than 0.5 the system starts to behave chaotically and becomes unpredictable.

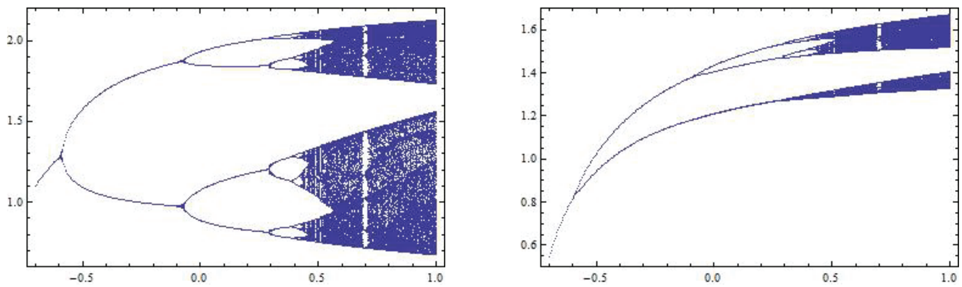

Figure 11: Bifurcation diagrams with respect to the parameter r against the variables q_1^* (left) and q_2^* (right) with 400 iterations of the map Eq.(16) for α = 5, c_1 = 1, c_2 = 1.3, f_1 = 0.5, f_2 = 0.7, β = 0.3, d = 0.5, μ = 0.5 and k = 0.76.

This chaotic behavior of the discrete dynamical system of Eq.(16) is visualized by strange attractors (Fig.13-left) that are created when the parameter r takes large values outside of the stability space as for example for r = 0.99. Also, Lyapunov numbers larger than 1.00 are calculated (Fig. 13-right) as an evidence for the chaotic behavior of the game's dynamical system.

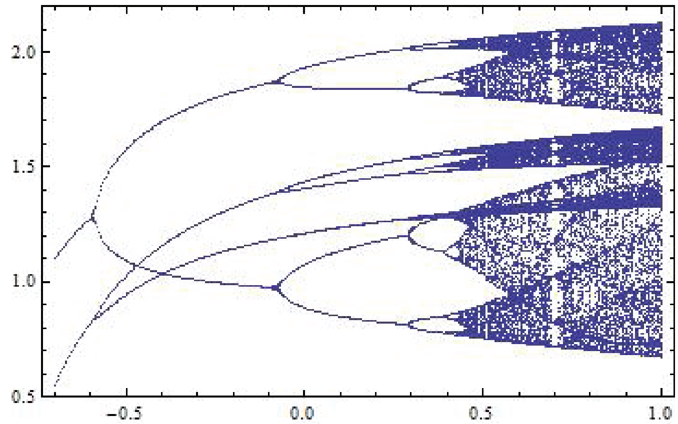

Figure 12: The two bifurcation diagrams of Fig.11 are plotted in one.

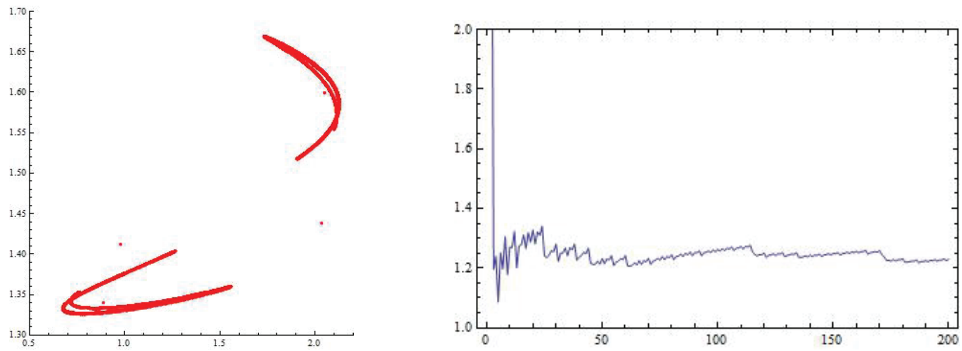

Figure 13: Phase portrait (strange attractor) (left) and Lyapunov numbers (right) of the orbit of (0.1,0.1) with 8000 iterations of the map Eq.(16) for $\alpha = 5$, $c_1 = 1$, $c_2 = 1.3$, $f_1 = 0.5$, $f_2 = 0.7$, $\beta = 0.3$, $d = 0.5$, $\mu = 0.5$, $k = 0.76$ and $r = 0.99$.

Finally, for these values of the parameter r (outside the stability space), make the system sensitive on its initial conditions. This means that a small change on the coordinates of initial conditions can make the system to have differentiations after a number of iterations. The time series of q_1- coordinate for two different initial conditions (0.1,0.1) (Fig.14-left) and (0.101,0.1) (Fig.14-right) are plotted. At the beginning the time series are indistinguishable, but after a number of iterations the difference between them builds up rapidly.

2.5. Chaos control

As it seems for values greater than 0.60 of the parameter k, our system slowly enters a chaotic behavior. In this section a try to control this behavior by introducing a new parameter $m \in (0,1)$ is presented. For some values of this parameter there is a stable Nash equilibrium and for every value of the parameter k that is outside the previous

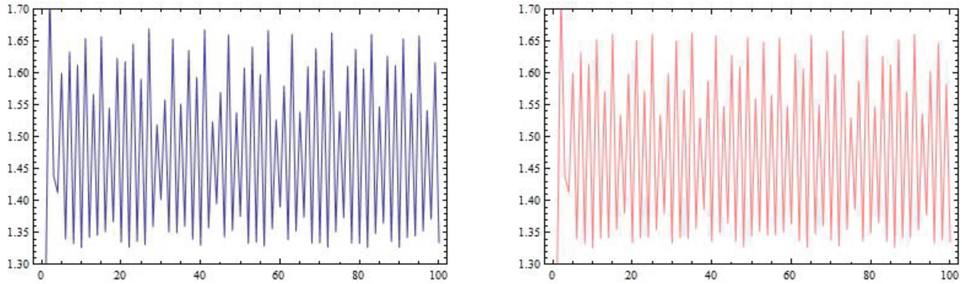

Figure 14: Sensitive dependence on initial conditions for q_1 -coordinate plotted against the time: the orbit of (0.1,0.1) (left) and the orbit of (0.101,0.1) (right) of the system Eq.(16) for $\alpha = 5$, $c_1 = 1$, $c_2 = 1.3$, $f_1 = 0.5$, $f_2 = 0.7$, $\beta = 0.3$, $d = 0.5$, $\mu = 0.5$, $k = 0.76$ and $r = 0.99$.

stability space. This parameter m introduced in discrete dynamical system of Eq.(16) as follows:

$$\begin{cases} q_1(t+1) = (1-m)\cdot\left[q_1(t) + k\cdot q_1(t)\cdot\frac{\partial U_1}{\partial q_1}\right] + m\cdot q_1(t) \\ q_2(t+1) = (1-m) \\ \cdot\left\{\beta\cdot\left(q_2(t) + k\cdot q_2(t)\cdot\frac{\partial U_2}{\partial q_2}\right) + (1-\beta)\cdot\frac{1}{2}\cdot\left(\alpha - c_2 - \frac{f_2}{1+r} - 2q_2 - d\cdot(1-\mu)\cdot q_1(t)\right)\right\} \\ +m\cdot q_2(t)\,(30) \end{cases}$$

The new stability conditions are formulated by the following proposition:

Proposition 2:

The Nash equilibrium position $E_* = \left(q_1^*, q_2^*\right)$ *of the discrete dynamical system of Eq.(30) is locally asymptotically stable if:*

$$2\beta\cdot k^2\cdot(1-m)^2\left[4-d^2\cdot(1-\mu)^2\right]\cdot q_1^*\cdot q_2^* - k\cdot(1-m)$$

$$\cdot\left[8\cdot(\beta+m-\beta\cdot m) - 4 + d^2\cdot(1-\mu)^2\cdot(1-\beta)\cdot(1-m)\right]\cdot q_1^* -$$

$$-4k\cdot\beta\cdot(1-m)\cdot q_2^* + 4(\beta+m-\beta\cdot m-1) < 0(31)$$

and

$$2\beta\cdot k^2\cdot(1-m)^2\left[4-d^2\cdot(1-\mu)^2\right]\cdot q_1^*\cdot q_2^* - k$$

$$\cdot(1-m)\cdot\left[8\cdot(\beta+m-\beta\cdot m) + d^2\cdot(1-\mu)^2\cdot(1-\beta)\cdot(1-m)\right]\cdot q_1^* -$$

$$-8k\cdot\beta\cdot(1-m)\cdot q_2^* + 8(\beta+m-\beta\cdot m) > 0(32)$$

2.5.1. Numerical simulations focusing on the parameter m

The Figure 15 shows the stability region between the parameters k (horizontal axis) and the parameter m (vertical axis) for the specific values of the other parameters as follows: $\alpha = 5$, $c_1 = 1$, $c_2 = 1.3$, $f_1 = 0.5$, $f_2 = 0.7$, $\beta = 0.3$, $d = 0.5$, $\mu = 0.5$ and $r = 0.1$. As it clearly seems this new control parameter m creates a local stability space in which this parameter allow the Nash Equilibrium to remain in local stability for every value of the parameter k (speed of adjustment).

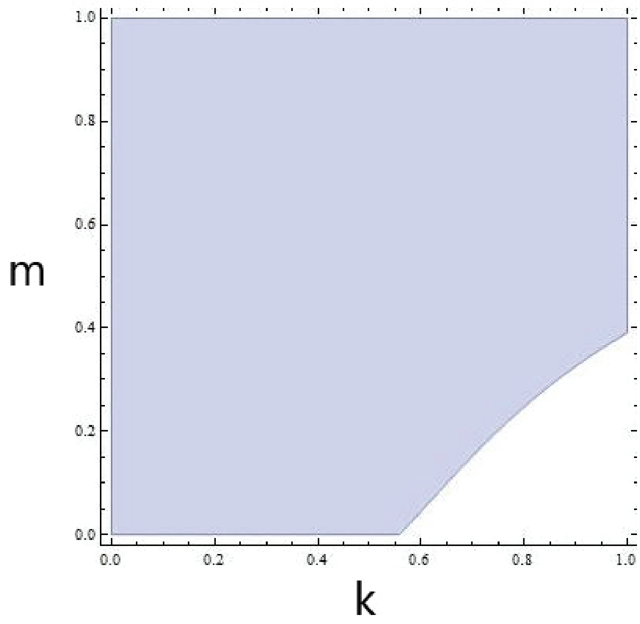

Figure 15: Stability region of the Nash Equilibrium of Eq.(30) for $\alpha = 5$, $c_1 = 1$, $c_2 = 1.3$, $f_1 = 0.5$, $f_2 = 0.7$, $\beta = 0.3$, $d = 0.5$, $\mu = 0.5$ and $r = 0.1$.

For example if the parameter k takes the value of 0.90 (outside of the primary stability space), then the local stability space focusing to the parameter m is m (0 , 0.30). This result is verified by the Figure ?? where the bifurcation diagrams against the production quantities q_1^* (left) and q_2^* (right) with respect to the parameter m are plotted for $\alpha = 5$, $c_1 = 1$, $c_2 = 1.3$, $f_1 = 0.5$, $f_2 = 0.7$, $\beta = 0.3$, $d = 0.5$, $\mu = 0.5$, $r = 0.1$ and $k = 0.9$ are shown. It is clear that as the values of the parameter m get larger and larger, after a limit of these values the Nash Equilibrium becomes locally asymptotically stable. As the parameter m takes values lower than 0.30, the E_* position lost its local stability and for even lower values of the parameter m, the discrete dynamical system becomes unstable and unpredictable through doubling period bifurcations.

Strange attractors are created and Lyapunov numbers larger than 1.00 are computed for small values of the parameter m outside the stability space. For example setting the

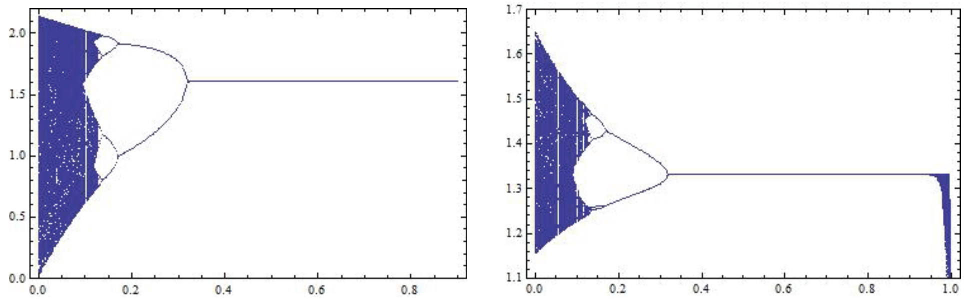

Figure 16: Bifurcation diagrams with respect to the parameter m against the variables q_1^* (left) and q_2^* (right) with 400 iterations of the map Eq.(30) for α = 5, c_1 = 1, c_2 = 1.3, f_1 = 0.5, f_2 = 0.7, β = 0.3, d = 0.5, μ = 0.5, r = 0.1 and k = 0.9.

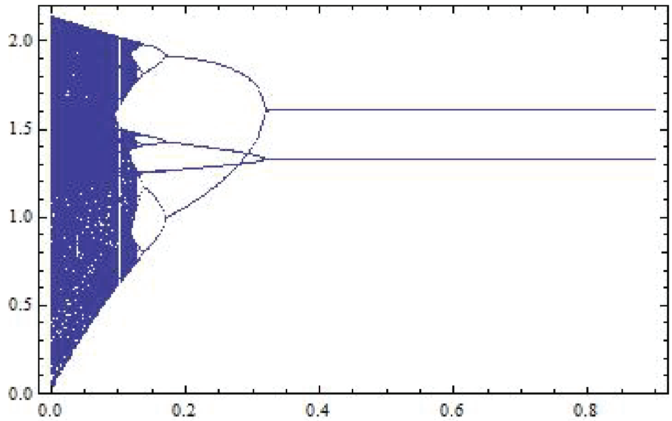

Figure 17: The two bifurcation diagrams of Fig.15 are plotted in one.

value of 0.05 to the parameter m and all other parameters have the same fixed values: α = 5, c_1 = 1, c_2 = 1.3, f_1 = 0.5, f_2 = 0.7, β = 0.3, d = 0.5, μ = 0.5, r = 0.1 and k = 0.9, the strange attractor of Figure 18 (left) is appeared and the Lyapunov numbers' diagram is plotted if Figure 18 (right). As it seems the Lyapunov numbers are clearly larger than 1.00 giving an evidence for chaos.

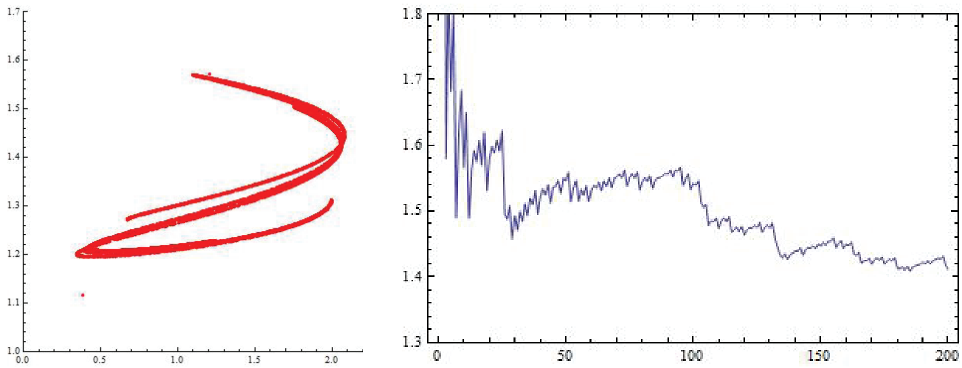

Figure 18: Phase portrait (strange attractor) (left) and Lyapunov numbers (right) of the orbit of (0.1,0.1) with 8000 iterations of the map Eq.(30) for α = 5, c_1 = 1, c_2 = 1.3, f_1 = 0.5, f_2 = 0.7, β = 0.3, d = 0.5, μ = 0.5, r = 0.1, k = 0.9 and m = 0.05.

Finally, the Figures 19 show that when the values of the parameter m are outside of its stability region, the dynamical system behaves chaotically. For these values of the parameter m, only a small change on the initial conditions can bring large differentiations to the system's behavior after a number of iterations.

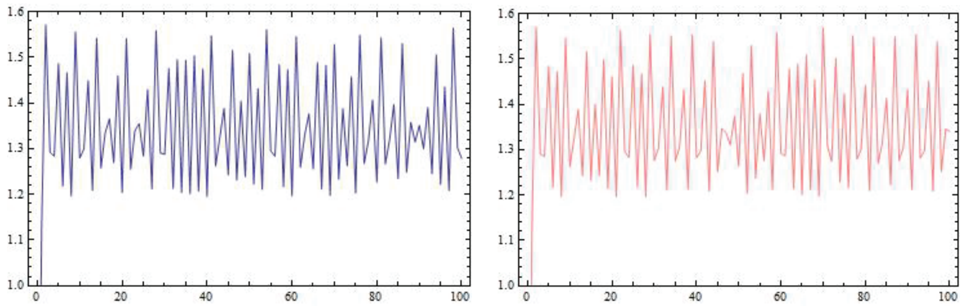

Figure 19: Sensitive dependence on initial conditions for q_1 -coordinate plotted against the time: the orbit of (0.1,0.1) (left) and the orbit of (0.101,0.1) (right) of the system Eq.(30) for $\alpha = 5$, $c_1 = 1$, $c_2 = 1.3$, $f_1 = 0.5$, $f_2 = 0.7$, $\beta = 0.3$, $d = 0.5$ $\mu = 0.5$, $r = 0.1$, $k = 0.9$ and $m = 0.05$.

3. CONCLUSION

This study, contains the dynamics of a nonlinear discrete-time Cournot-type duopoly game, where the players have heterogeneous expectations based on the marginal utilities through a discrete dynamical system. The stability of equilibrium points, bifurcations and chaotic behavior are investigated. It is proved that higher values of the parameters k (speed of adjustment), β (second player's adaptation probability) and r (percentage domestic energy adequacy) may destabilize the local stability of Nash equilibrium position and cause a chaotic behavior for the system, through period-doubling bifurcation. The chaotic features are justified numerically via presenting Lyapunov numbers, strange attractors and sensitive dependence on initial conditions. This chaotic structure controlled introducing a new parameter m. A new stability space for the control parameter m is found, in which the Nash equilibrium becomes locally asymptotically stable of every value of the parameter k. Finally, a useful property is shown that for small values of the speed of adjustment the Nash equilibrium is locally asymptotically stable for every value of the adaptation parameter β.

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