Conference Paper

# On a Bertrand Duopoly Game With Heterogeneous Expectations and Emissions Costs

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## Abstract

This study was based on the dynamics of a nonlinear Bertrand-type duopoly game with differentiated goods, linear demand and a cost function that included emissions costs. This duopoly game was modeled with a system of two difference equations. Existence and stability of equilibria of this system were studied. It was shown that the model gave more complex chaotic and unpredictable trajectories as a consequence of changes in the speed of adjustment parameter and horizontal product differentiation parameter. Numerical simulations showed that a higher value of the speed of adjustment and a higher or lower (negative) value of product differentiation (weaker or fiercer competition) can destabilize the economy. The chaotic features were justified numerically via computing Lyapunov numbers and sensitive dependence on initial conditions. Also, it was shown that in this case of a duopoly game, there were stable trajectories, and a higher (lower) degree of product differentiation did not tend to destabilize the economy.

**Keywords:** Bertrand duopoly game, discrete dynamical system, heterogeneous expectations, stability, chaotic behavior

## 1. Introduction

Joseph Louis Francois Bertrand, the French mathematician in 1883 modified Cournot's game suggesting that the players (sellers) actually choose prices rather the quantities. Bertrand model originally is based on the premise that all players take decisions by naïve way, so that in every step, each player assumes the last values were taken by the competitors without an estimation of their future reactions. However, under the conditions of real market, such an assumption is very unlikely since not all players share naïve beliefs. Different approaches to firm behavior were proposed. Some of the authors considered duopolies under homogeneous expectations and found a variety of complex dynamics in their games, such as appearance of strange attractors (Agiza 1999; Agiza et al. 2002; Agliari et al. 2005, 2006; Bischi and Kopel 2001; Kopel 1996; Puu 1998, 2005; Sarafopoulos 2015a,b; Zhang et al. 2009). Also, models with heterogeneous agents were studied (Agiza and Elsadany 2003, 2004; Agiza et al. 2002; Den Haan 2001; Hommes 2006; Fanti and Gori 2012; Gao 2009; Sarafopoulos and Papadopoulos 2017, 2019, 2020; Tramontana 2010; Zhang et al. 2007; Wu et al. 2010).

In real market, producers do not know the entire demand function, though it is possible that they have a perfect knowledge of technology, represented by the cost function. Here it is more likely that firms employ some local estimate of the demand. This issue has been previously analyzed (Baumol and Quandt 1964; Singh Vives 1984; Puu 1991, 1995; Westerhoff 2006; Naimzada and Ricchiuti 2008; Askar 2013, 2014). Bounded rational players (sellers) update their strategies based on discrete time periods and by using a local estimate of the marginal profit. With such local adjustment mechanism, the players are not requested to have a complete knowledge of the demand and the cost functions (Agiza and Elsadany 2004; Elsadany and Awad 2016; Naimzada and Sbragia 2006; Zhang et al. 2007; Askar 2014).

The present paper is a partial approach to our main ongoing research objective, which is the emergence of complexity in various oligopoly models as well as its control. In this study the dynamics of a Bertrand-type duopoly game with differentiated goods where each firm behaves with heterogeneous expectation strategies using cost functions that contain emission costs are studied. It is shown that the model gives more complex chaotic and unpredictable trajectories as a consequence of change in the bounded rational player's speed of adjustment and the parameter of horizontal product differentiation. The paper is organized as follows: Section 2 includes the construction of this game (subsection 2.1) exporting the discrete dynamical system in which the dynamical analysis based on. Also, in the same section (subsection 2.2) the dynamics of the duopoly game with heterogeneous expectations, linear demand and cost function including emission costs (Sarafopoulos and Papadopoulos 2017) are developed proving the equilibrium points' existence and local stability. In subsections 2.2.3 and 2.2.4 the complexity of game's discrete dynamical system are revealed via computing Lyapunov numbers, strange attractors and sensitive dependence on initial conditions using numerical simulations for specific values of the game's parameters.

## 2. The game

### 2.1. The construction of the game

In this paper heterogeneous players are considered and more specifically, it is considered that the Firm 1 chooses the price of its product in a rational way, following an adjustment mechanism (bounded rational player), while the Firm 2 decides with naïve way by selecting a price that maximizes its output (naïve player). A simple Bertrand-type duopoly market is assumed, where firms (players) produce differentiated goods and offer them at discrete-time periods on a common market. Price decisions are taken at discrete time periods t = 0, 1, 2,... At each period t, every firm must form an expectation of the rival's strategy in the next time period in order to determine the corresponding profit-maximizing prices for period t+1. It supposed that $q_1$, $q_2$ are the production quantities of each firm. Also, the hypothesis that the preferences of consumers represented by the following equation is used:

$$U\left(q_1, q_2\right) = \alpha\left(q_1 + q_2\right) - \frac{1}{2}\left(q_1^2 + q_2^2 + 2d \cdot q_1 q_2\right) \quad (1)$$

where the positive parameter $\alpha > 0$ expresses the market size and the parameter $d \in (-1, 1)$ reveals the differentiation degree between two players' products. For example, if d = 0, then both products are independent and each firm participates in a monopoly market. When the parameter d takes the maximum positive value and d = 1, then each product is a substitute product for the other, since the products are homogeneous. It is understood that for positive values of the parameter d the larger the value, the less diversification there is between two products. On the other hand, negative values of the parameter d describe that the both products are complementary and when this parameter takes the minimum negative value and d = - 1, then the phenomenon of full competition between the two companies is appeared. The inverse demand functions (as functions of quantities) coming from the maximizing of Eq.(1) are given by the following equations:

$$p_1\left(q_1, q_2\right) = \alpha - q_1 - d \cdot q_2 \text{ and } p_2\left(q_1, q_2\right) = \alpha - q_2 - d \cdot q_1 \tag{2}$$

Calculating the direct demand functions (as functions of prices $p_1$ and $p_2$) it gives the equations:

$$q_1\left(p_1, p_2\right) = \frac{\alpha(1-d) - p_1 + d \cdot p_2}{1-d^2} \tag{3}$$

and

$$q_2\left(p_1, p_2\right) = \frac{\alpha(1-d) - p_2 + d \cdot p_1}{1-d^2} \tag{4}$$

It is supposed the following linear cost function for each i firm (i = 1,2):

$$C_i\left(q_i\right) = C_{p_i} + C_{e_i} \tag{5}$$

where

$$C_{pi}\left(q_i\right) = c \cdot q_i \tag{6}$$

is the clear linear production cost for each firm using the same marginal production cost $c > 0$ and:

$$C_{ei}\left(q_i\right) = p_c \cdot \epsilon \cdot q_i \tag{7}$$

are the costs due to the fumes emissions of the production process, where $p_c$ is the price of emission license that is announced by the Government, $\epsilon \in [0, 1]$ is the common nonnegative coefficient for two firms and multiplying with the payers' productions is gives the total emissions.

With the above assumptions the profit functions of two firms are calculated as follows:

$$\Pi_1\left(p_1, p_2\right) = \frac{\left(p_1 - c - p_c \cdot \epsilon\right)\left[\alpha(1-d) - p_1 + d \cdot p_2\right]}{1-d^2} \tag{8}$$

and

$$\Pi_2\left(p_1, p_2\right) = \frac{\left(p_2 - c - p_c \cdot \epsilon\right)\left[\alpha(1-d) - p_2 + d \cdot p_1\right]}{1-d^2} \tag{9}$$

As a result the marginal profits for two players are given by the equations:

$$\frac{\partial \Pi_1}{\partial p_1} = \frac{1}{1-d^2} \cdot \left[\alpha (1-d) + c + p_c \cdot \epsilon - 2p_1 + d \cdot p_2\right] \tag{10}$$

and

$$\frac{\partial \Pi_2}{\partial p_2} = \frac{1}{1-d^2} \cdot \left[\alpha (1-d) + c + p_c \cdot \epsilon - 2p_2 + d \cdot p_1\right] \tag{11}$$

The final stage to construct the game's discrete dynamical system contain the requirement of the players expectations, where as mentioned above the first firm is characterized as bounded rational player who decides to increase his level of adaptation in a mechanism if he has a positive marginal profit, or decrease his level if the marginal profit is negative. According to the existing literature this mechanism described by the following dynamical equation:

$$\frac{p_1 (t+1) - p_1 (t)}{p_1 (t)} = k \cdot \frac{\partial \Pi_1}{\partial p_1} \tag{12}$$

where the positive parameter k > 0, expresses the speed of adjustment of first firm and it gives the extent $p_1$ price variation of the firm following a given profit signal. Moreover it captures the fact that relative variations of the price are proportional to the marginal profit. On the other hand, the second firm decides by a naïve way, selecting this $p_2$ price that maximizes its profits (naïve player):

$$\frac{\partial \Pi_2}{\partial p_2} = 0 \quad \Rightarrow \quad p_2 (t+1) = \frac{\alpha (1-d) + c + p_c \cdot \epsilon + d \cdot p_1 (t)}{2} \tag{13}$$

The game's dynamical system for these two players is described by:

$$\begin{cases} p_1 (t+1) = p_1 (t) + k \cdot p_1 (t) \cdot \dfrac{\partial \Pi_1}{\partial p_1} \\ p_2 (t+1) = \dfrac{\alpha (1-d) + c + p_c \cdot \epsilon + d \cdot p_1 (t)}{2} \end{cases} \tag{14}$$

This study focuses on the dynamics of this system of Eq(14) with respect to the parameters k and d.

## 2.2. Dynamical analysis

### 2.2.1. The equilibriums of the game

The equilibrium positions of the dynamical system Eq.(14) are resulted by the nonnegative solutions of the algebraic system:

$$\begin{cases} k \cdot p_1^* \cdot \dfrac{1}{1-d^2} \cdot \left[\alpha (1-d) + c + p_c \cdot \epsilon - 2p_1^* + d \cdot p_2^*\right] = 0 \\ p_2^* = \dfrac{\alpha (1-d) + c + p_c \cdot \epsilon + d \cdot p_1^*}{2} \end{cases} \tag{15}$$

which obtained by setting $p_1 (t+1) = p_1 (t) = p_1^*$ and $p_2 (t+1) = p_2 (t) = p_2^*$ in Eq. (14).

If $p_1^* = 0$, then $p_2^* = \dfrac{\alpha(1-d) + c + p_c \cdot \epsilon}{2}$ and these are the coordinates of the boundary equilibrium:

$$E_0 = \left(0, \frac{\alpha(1-d) + c + p_c \cdot \epsilon}{2}\right) \tag{16}$$

If $\dfrac{\partial \Pi_1}{\partial p_1} = \dfrac{\partial \Pi_2}{\partial p_2} = 0$, then the system is formed:

$$\begin{cases} p_1^* = \dfrac{\alpha(1-d) + c + p_c \cdot \epsilon + d \cdot p_2^*}{2} \\ p_2^* = \dfrac{\alpha(1-d) + c + p_c \cdot \epsilon + d \cdot p_1^*}{2} \end{cases} \tag{17}$$

The previous system's solutions are:

$$p_1^* = p_2^* = \frac{\alpha(1-d) + c + p_c \cdot \epsilon}{2-d}$$

giving the game's Nash equilibrium:

$$E_* = \left(\frac{\alpha(1-d) + c + p_c \cdot \epsilon}{2-d}, \frac{\alpha(1-d) + c + p_c \cdot \epsilon}{2-d}\right) \tag{18}$$

### 2.2.2. Stability of equilibriums

The study of the local stability of the equilibrium is based on the localization on the complex plane of the eigenvalues of the Jacobian matrix of the dimensional map (Eq.(14)). In order to study the local stability of equilibrium points of the model (14), the Jacobian matrix $J(p_1, p_2)$ along the variable strategy $(p_1, p_2)$ is considered:

$$J(p_1, p_2) = \begin{bmatrix} f_{p_1} & f_{p_2} \\ g_{p_1} & g_{p_2} \end{bmatrix} \tag{19}$$

Where

$$\begin{aligned} f(p_1, p_2) &= p_1 + k \cdot p_1 \cdot \frac{\partial \Pi_1}{\partial p_1} \\ g(p_1, p_2) &= \frac{\alpha(1-d) + c + p_c \cdot \epsilon + d \cdot p_1}{2} \end{aligned} \tag{20}$$

The Jacobian matrix is:

$$J(p_1, p_2) = \begin{bmatrix} 1 + \dfrac{k}{1-d^2} \cdot \left[\alpha(1-d) + c + p_c \cdot \epsilon - 4p_1 + d \cdot p_2\right] & \dfrac{k \cdot d \cdot p_1}{1-d^2} \\ \dfrac{d}{2} & 0 \end{bmatrix} \tag{21}$$

At the equilibrium $E_0$:

$$J(E_0) = \begin{bmatrix} 1 + \dfrac{k}{1-d^2} \cdot \dfrac{(2+d)\left[\alpha(1-d) + c + p_c \cdot \epsilon\right]}{2} & 0 \\ \dfrac{d}{2} & 0 \end{bmatrix} \tag{22}$$

with

$$Tr = 1 + \frac{k}{1-d^2} \cdot \frac{(2+d)\left[\alpha(1-d)+c+p_c\cdot\epsilon\right]}{2} \text{ and } Det = 0.$$

The characteristic equation of $J\left(E_0\right)$ is:

$$l^2 - Tr\cdot l + Det = 0 \tag{23}$$

with solutions

$$l_1 = 0$$

and

$$l_2 = Tr = 1 + \frac{k}{1-d^2} \cdot \frac{(2+d)\left[\alpha(1-d)+c+p_c\cdot\epsilon\right]}{2} \tag{24}$$

Since $l_2 > 1$, the equilibrium $E_0$ is unstable.

At the Nash equilibrium point $E_*$ the Jacobian matrix is:

$$J\left(E_*\right) = \begin{bmatrix} 1 + \frac{k}{1-d^2}\cdot\left[\alpha(1-d)+c+p_c\cdot\epsilon-(4-d)\cdot p^*\right] & \frac{k\cdot d\cdot p^*}{1-d^2} \\ \frac{d}{2} & 0 \end{bmatrix} \tag{25}$$

with

$$Tr = 1 - \frac{2k\cdot p^*}{1-d^2} \text{ and } Det = -\frac{k\cdot d^2\cdot p^*}{2\left(1-d^2\right)}. \tag{26}$$

The Nash equilibrium is asymptotically stable if the following conditions are hold (Gandolfo 1997; Elaydi 2005):

$$\begin{aligned} &i) \quad 1 - Det > 0 \\ &ii) \quad 1 - Tr + Det > 0 \\ &iii) \quad 1 + Tr + Det > 0 \end{aligned} \tag{27}$$

Since

$$1 - Det = 1 + \frac{k\cdot d^2\cdot p^*}{2\left(1-d^2\right)} > 0 \tag{28}$$

and

$$1 - Tr + Det = \frac{k\cdot p^*\cdot\left(4-d^2\right)}{2\left(1-d^2\right)} > 0 \tag{29}$$

the conditions (i) and (ii) are always satisfied.

The third condition becomes as:

$$1 + Tr + Det > 0 \quad \Leftrightarrow \quad k\cdot p^*\cdot\left(4+d^2\right) - 4\left(1-d^2\right) < 0 \tag{30}$$

with

$$p^* = \frac{\alpha(1-d) + c + p_c \cdot \epsilon}{2-d}$$

Then

$$k \cdot \left[\alpha(1-d) + c + p_c \cdot \epsilon\right] \cdot \left(4 + d^2\right) - 4(2-d)\left(1-d^2\right) < 0 \quad \Leftrightarrow$$

$$-(4+\alpha k) \cdot d^3 + \left[\left(\alpha + c + p_c \cdot \epsilon\right) \cdot k + 8\right] \cdot d^2 +$$

$$+4(1-\alpha k) \cdot d + 4k \cdot \left(\alpha + c + p_c \cdot \epsilon\right) - 8 < 0 \tag{31}$$

**Proposition:**

*The Nash equilibrium* $E_*\left(p_1^*, p_2^*\right)$ *of the dynamical system Eq.(14) is locally asymptotically stable if:*

$$-(4+\alpha k) \cdot d^3 + \left[\left(\alpha + c + p_c \cdot \epsilon\right) \cdot k + 8\right] \cdot d^2 +$$

$$+4(1-\alpha k) \cdot d + 4k \cdot \left(\alpha + c + p_c \cdot \epsilon\right) - 8 < 0$$

Numerical simulations focusing on the parameter k

To provide some numerical evidence for the chaotic behavior of the system Eq.(14), as a consequence of change in the parameter k of the speed of adjustment of player 1 and the parameter d of the product differentiation degree, various numerical results are presented here to show the chaoticity, including two dimensional stability space between these two parameters, bifurcations diagrams, strange attractor, Lyapunov numbers and sensitive dependence on initial conditions (Kulenovic, M., Merino, O., 2002). In order to study the local stability properties of the equilibrium points, it is convenient to set some fixed values to the parameters $\alpha,\ c,\ p_c\ and\ \epsilon$ for example as follows: $\alpha = 5,\ c = 1,\ p_c = 0.5,\ \epsilon = 0.4$ and the stability space between the parameters k and d is plotted (Fig.1).

Focusing on the parameter k the stability condition can be written as:

$$k < \frac{4d^3 - 8d^2 - 4d + 8}{\left(4 + d^2\right)\left[\alpha(1-d) + c + p_c \cdot \epsilon\right]} \tag{32}$$

Using the Eq.(18) it seems that the Nash Equilibrium position does not depend on the parameter k and as a result setting some specific values to the other parameters, it is a fixed point. For example choosing the specific values for the parameters: α = 5, c = 1, $p_c$ = 0.5, ε = 0.4 and d = 0.5 the Nash Equilibrium position is the point:

$$E_* = (2.47, 2.47)$$

and the stability condition becomes as:

$$0 < k < 0.29$$

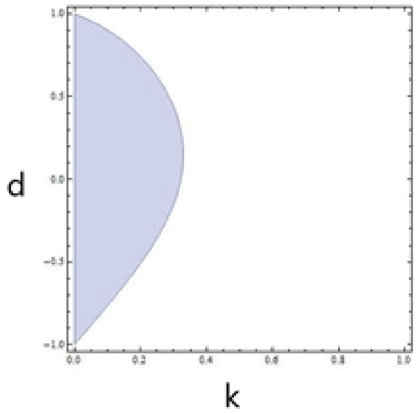


**Figure** 1: Stability space between the parameters k (horizontal axis) and the parameter d (vertical axis) for $\alpha = 5, \quad c = 1, \quad p_c = 0.5, \quad \epsilon = 0.4.$

This result is verified by the bifurcation diagrams (Fig.2) against the variables $p_1$(left) and $p_2$(right) with respect to the parameter k. Also in Figure 3 these two bifurcation diagrams of Fig.2 in one are plotted to show the common stability space that appears for the parameter k. In these figures the Nash equilibrium E* is locally asymptotically stable for 0 < k < 0.29 verifying the previous algebraic result. For d < 0.29 the Nash equilibrium E* becomes unstable, and one observes complex dynamics behavior such as cycles of higher order and chaos.

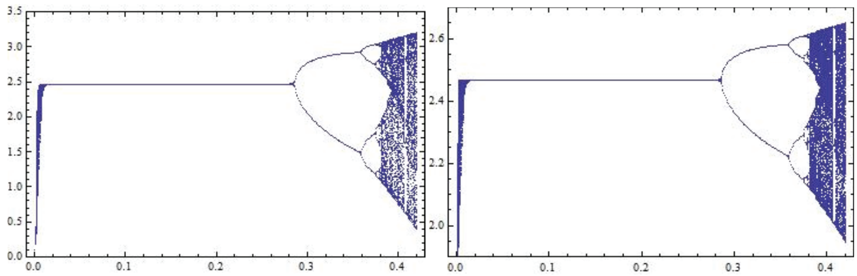

**Figure** 2: Bifurcation diagrams with respect to the parameter d against the variables $p_1^*$ (left) and $p_2^*$ (right) with 400 iterations of the map Eq.(14) for $\alpha = 5, \quad c = 1, \quad p_c = 0.5, \quad \epsilon = 0.4, \quad k = 0.3.$

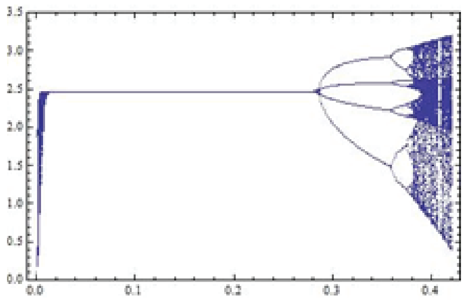

**Figure** 3: Two bifurcation diagrams of Fig.2 are plotted in one.

Figure 4 shows the graph of the orbit of the point (0.1,0.1) (strange attractor) and Lyapunov numbers for $\alpha = 5, \quad c = 1, \quad p_c = 0.5, \epsilon = 0.4, \quad d = 0.5$ and k = 0.42. From these results when all parameters are fixed and only k is varied the structure of the game becomes complicated through period doubling bifurcations, more complex bounded attractors are created which are aperiodic cycles of higher order or chaotic attractors. On more evidence for chaos are the Lyapunov number (Fig.4 (right)) that they are greater of 1.

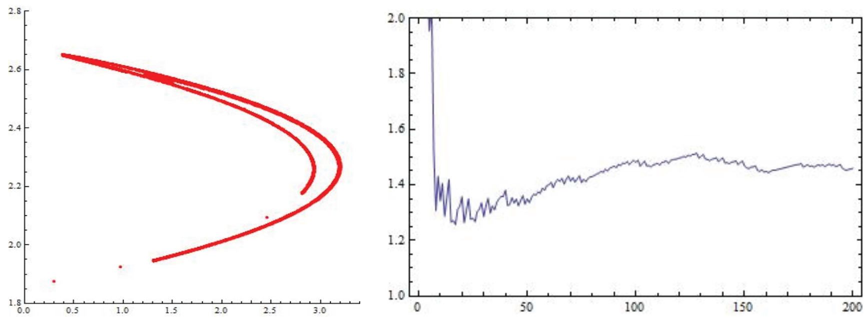


**Figure** 4: Phase portrait (strange attractor) (left) and Lyapunov numbers (right) of the orbit of (0.1,0.1) with 8000 iterations of the map Eq.(14) for $\alpha = 5, \ \ c = 1, \ \ p_c = 0.5, \ \ \epsilon = 0.4, \ \ d = 0.5$ and k = 0.42.

To demonstrate the sensitivity to initial conditions of the system Eq.(14) we compute two orbits with initial points (0.1,0.1) and (0.101,0.1), respectively. Figure 5 shows the sensitive dependence on initial conditions for $p_1$-coordinate of the two orbits, for the system Eq.(14), plotted against the time with the parameter values $\alpha = 5, \ \ c = 1, \ \ p_c = 0.5, \ \ \epsilon = 0.4, \ \ d = 0.5$ and k = 0.42. At the beginning the time series are indistinguishable; but after a number of iterations, the difference between them builds up rapidly, which is clearly shown in Figure 6. From Figures 5 and 6 we show that the time series of the system Eq.(14) is sensitive dependence to initial conditions, i.e. complex dynamics behavior occur in this model.

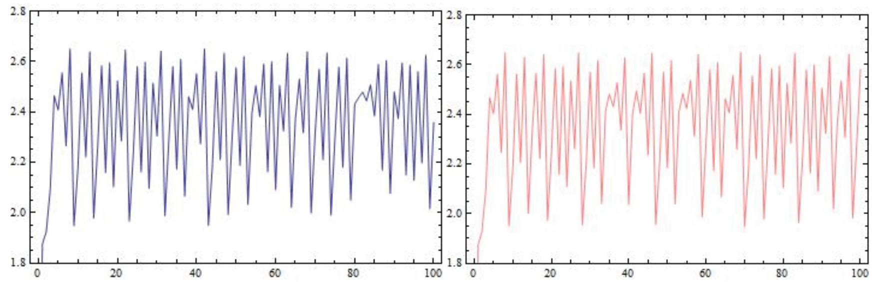

**Figure** 5: Sensitive dependence on initial conditions for $p_1$-coordinate plotted against the time: the orbit of (0.1,0.1) (left) and the orbit of (0.101,0.1) (right) of the system Eq.(14) for $\alpha = 5, \ \ c = 1, \ \ p_c = 0.5, \ \ \epsilon = 0.4, \ \ d = 0.5$ and $k = 0.42$.

### 2.2.3. Numerical simulations focusing on the parameter d

In this section some numerical evidence for the chaotic behavior of the system Eq.(14), as a consequence of change in the parameter d of the product differentiation degree are provided. In order to study the local stability properties of the equilibrium points, fixed values for the parameters $\alpha, \ \ c, \ \ p_c, \ \ \epsilon$ and $k$are taken. For example the following

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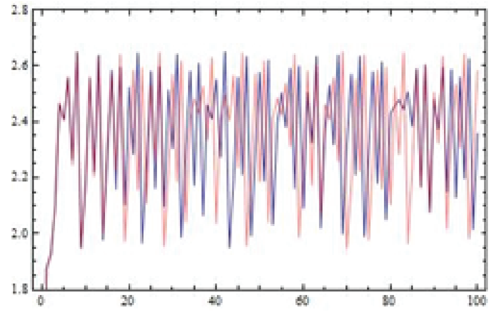


**Figure** 6: Two bifurcation diagrams of Fig.6 are plotted in one.

values of these parameters are chosen: $\alpha = 5, \ \ c = 1, \ \ p_c = 0.5, \ \ \epsilon = 0.4, \ k = 0.3$ and the Eq.(30) becomes as:

$$-5.5 \cdot d^3 + 9.56 \cdot d^2 - 2d - 0.56 < 0$$

and its graph is presented at the Figure 7, in which it seems that it becomes negative for values of parameter d from about -0.15 until 0.5. This means that when d is between these values the Nash equilibrium of the system Eq.(14) is stable.

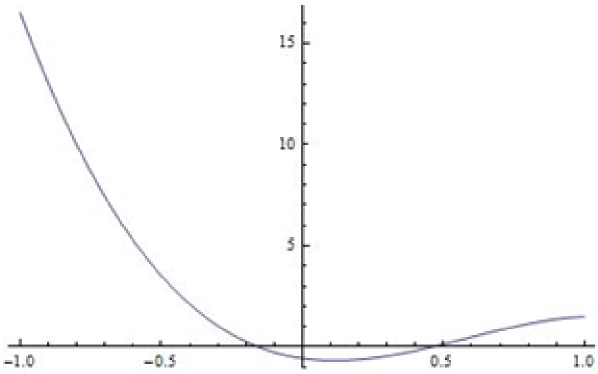


**Figure** 7: Graph of the function: $f(d) = -5.5 \cdot d^3 + 9.56 \cdot d^2 - 2d - 0.56$ , with $d \in [-1, 1]$.

Numerical experiments are computed to show the bifurcation diagram with respect to d, strange attractor of the system Eq.(14) in the phase plane $(p_1, p_2)$, and Lyapunov numbers. In figures 8 and 9 the bifurcation diagrams with respect to the parameter d against variable $p_1$(left) and $p_2$(right) are presented. In these figures the Nash equilibrium E* is locally asymptotically stable for -0.15 < d < 0.5. For d > 0.5 and d < -0.15 the Nash equilibrium E* becomes unstable, and one observes complex dynamics behavior such as cycles of higher order and chaos.

Figure 10 shows the graphs of the orbit of the point (0.1,0.1) (strange attractors) for $\alpha = 5, \ c = 1, \ p_c = 0.5, \ \epsilon = 0.4, \ k = 0.3$ and d = -0.50 (left), and d = 0.75 (right). From these results when all parameters are fixed and only d is varied the structure of the game becomes complicated through period doubling bifurcations, more complex bounded attractors are created which are aperiodic cycles of higher order or chaotic attractors.

Figure 11 shows the Lyapunov numbers' diagram of the same orbit for $\alpha = 5, \ c = 1, \ p_c = 0.5, \ \epsilon = 0.4, \ k = 0.3$ and d = 0.75. If the Lyapunov number is greater of 1, one has evidence for chaos.

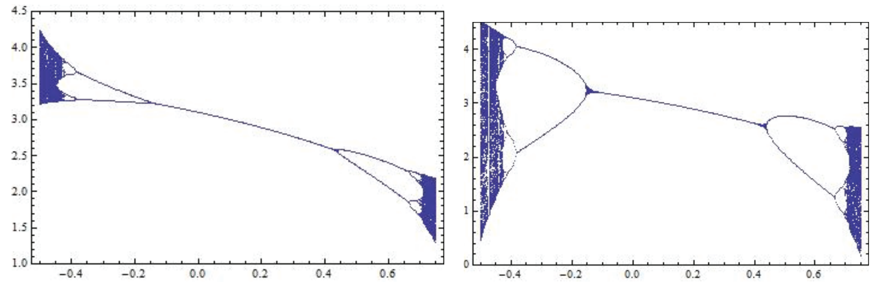


**Figure** 8: Bifurcation diagrams with respect to the parameter d against the variables $p_1^*$ (left) and $p_2^*$ (right) with 400 iterations of the map Eq.(14) for $\alpha = 5, \ \ c = 1, \ \ p_c = 0.5, \ \ \epsilon = 0.4, \ \ k = 0.3.$

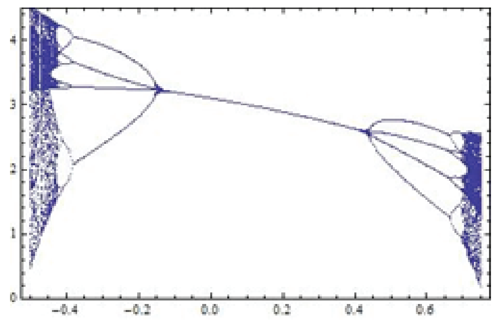

**Figure** 9: Two bifurcation diagrams of Fig.8 are plotted in one.

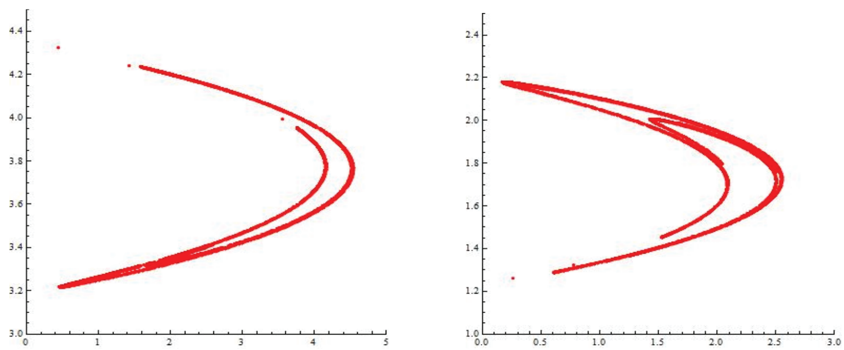

**Figure** 10: Phase portrait (strange attractor) of the orbit of (0.1,0.1) with 8000 iterations of the map Eq.(14) for $\alpha = 5, \ \ c = 1, \ \ p_c = 0.5, \ \ \epsilon = 0.4, \ \ k = 0.3$ and d = -0.50 (left) and d = 0.75 (right).

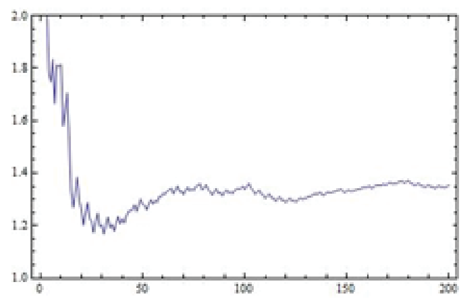

**Figure** 11: Lyapunov numbers (right) of the orbit of (0.1,0.1) with 8000 iterations of the map Eq.(14) for $\alpha = 5, \ \ c = 1, \ \ p_c = 0.5, \ \ \epsilon = 0.4, \ \ k = 0.3$ and d = 0.75.

To demonstrate the sensitivity to initial conditions of the system Eq.(14) we compute two orbits with initial points (0.1,0.1) and (0.101,0.1), respectively. Figure 6 shows sensitive dependence on initial conditions for $p_1$-coordinate of the two orbits, for the system

Eq.(14), plotted against the time with the parameter values $a = 5, \;\; c = 1, \;\; p_c = 0.5, \;\; \epsilon = 0.4, \;\; k = 0.3$ and d = 0.75. At the beginning the time series are indistinguishable; but after a number of iterations, the difference between them builds up rapidly, which is clearly shown in Figure 13. From Figures 12 and 13 we show that the time series of the system Eq.(14) is sensitive dependence to initial conditions, i.e. complex dynamics behavior occur in this model.

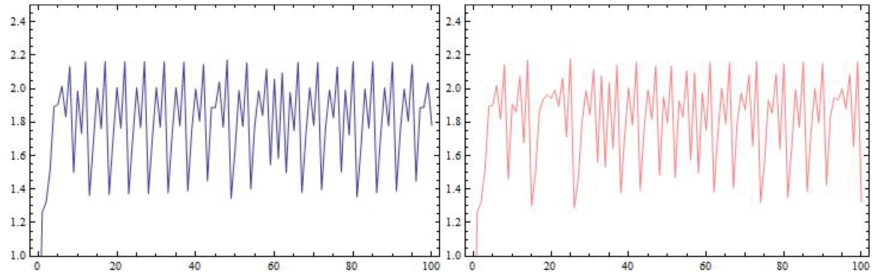

**Figure** 12: Sensitive dependence on initial conditions for $p_1$-coordinate plotted against the time: the orbit of (0.1,0.1) (left) and the orbit of (0.101,0.1) (right) of the system Eq.(14) for $\alpha = 5, \;\; c = 1, \;\; p_c = 0.5, \;\; \epsilon = 0.4, \;\; k = 0.3$ and $d = 0.75$.

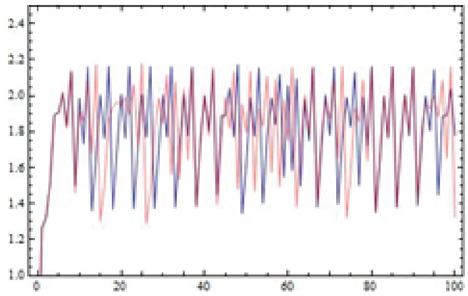

**Figure** 13: Two bifurcation diagrams of Fig.12 are plotted in one.

## 3. Conclusions

The present paper is a partial approach to our main ongoing research objective, which is the emergence of complexity in various oligopoly models as well as its control. In this study, through a discrete dynamical system based on the marginal profits of the players, the dynamics of a nonlinear discrete-time Bertrand-type duopoly game, where the players have heterogeneous expectations are studied. The stability of equilibrium points, bifurcations and chaotic behavior are investigated. It is proved that higher values of the speed of adjustment of bounded rational player and higher positive or lower negative values of product differentiation degree may change the stability of Nash equilibrium and cause a structure to behave chaotically, through period-doubling bifurcation. The chaotic features are justified numerically via computing Lyapunov numbers, strange attractors and sensitive dependence on initial conditions.

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