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On the Stability of a Cournot Dynamic Game Under the Influence of Information

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Abstract

In this paper we study the impact of information on the stability of a dynamic Cournot – type duopoly game. We suppose that one player searches for information about his rival before he makes his decision. We study how the amount of information acquired by player, influences the stability of Nash equilibrium. The game is modeled with a system of two difference equations. Existence and stability of equilibrium of this system are studied. We show numerically that the model gives chaotic and unpredictable trajectories as a consequence of change in the information parameter, but in our case there are also stable trajectories for each value of the information parameter. To provide some numerical evidence for the chaotic behavior of the system we present various numerical results including bifurcation diagrams, strange attractors, Lyapunov numbers and sensitive dependence on initial conditions.

Keywords: Cournot dynamic game, Nash equilibrium, Stability, Chaos.

1. Introduction

Oligopoly is a market between monopoly and perfect competition. Cournot the, French economist, firstly proposed a duopoly model choosing output as a decision variable in1838. He treated the case with naive expectations, so that in every step each player (firm) assumes the last values that were taken by the competitors without estimation of their future reactions. In 1950, the American mathematician Nash put forward the equilibrium theory of non-cooperative game, which provided a significant tool for the research of the oligopoly model. Subsequently, the Nash equilibrium point was found in both Cournot and Bertrand duopoly models. The game theory is applied to the oligopoly model which marks the first major development of the oligopoly model. However, the static equilibrium analysis of the system evolution is bound to lose a lot of valuable information. Only in the dynamic state can we observe the essence of the system evolution.

Since information in the market are usually incomplete, expectations play an important role. For instance, if firms do not know the output of the concurrent firm in advance, they are not able to compute the output that maximizes their profits and then every firm can adopt various mechanisms of expectations formation about the quantity offered by the rival. Some authors considered duopolies with homogeneous expectations (Agiza [1], Agiza *et al.* [4], Agliari *et al.* [5], [6], Bischi and Kopel [8], Kopel [15], Puu, [18], Sarafopoulos [21]). Also models with heterogeneous agents were studied (Agiza and Elsadany [2], [3], Agiza *et al*. [4], Den Haan [10], Fanti and Gori, [12], Tramontana [22], Zhang *et al*, [24]).

In this paper we consider heterogeneous firms in the sense that they are assumed to adopt different mechanisms to decide the output in each time period. In particular, we assume the following expectations: firm 1 (2) has bounded rational (naïve) expectations about the quantity to be produced in the future by the rival. Bounded rationality implies that the firm increases or decreases its output according to the information given by marginal profits obtained in the last period depending on a certain degree or intensity of reaction (Agiza and Elsadany [2], Naimzada and Sbragia [17], Zhang *et al.* [24], Askar [7]). This adjustement mechanism with respect to which firms decide to increase (decrease) the production if marginal profits are positive (negative), has been suggested and called "myopic" by Dixit (1986). In contrast with the first one, the second firm is a naïve player in the sense that it expects that rival will produce in the future a quantity equal to those produced in the last period. This adjustment mechanism dates back to the first formal theory of oligopoly by Cournot.

Information plays an important role in decision making process, firms in the market, believing that they can make a more rational decision if they master more information, are trying to acquire the information about both rival and market. Information acquisition is emphasized by business enterprises. The impacts of the information have been studied in many papers (Junhai and Zhanbing [14] and their references).

In this paper we study the impact of information on the stability of a dynamic Cournot – type duopoly game with differentiated products. We study how the information parameter influences the stability of Cournot-Nash equilibrium and we show that this parameter may destabilize this equilibrium. It may also cause unpredictable market fluctuations. Moreover, from a mathematical point of view, we show numerically that the destabilization of the fixed point can occur through a flip bifurcation and also that a cascade of flip bifurcations may lead to periodic cycles and deterministic chaos.

To provide some numerical evidence for the chaotic behavior of the system we present various numerical results including bifurcation diagrams, strange attractors, Lyapunov numbers and sensitive dependence on initial conditions. We show also that in our case there are stable trajectories for each value of the information parameter. Therefore, under certain conditions, this parameter does not tend to destabilize the market.

The paper is organized as follows. Section 2 presents the two-dimensional discrete dynamic system of a duopoly game with heterogeneous expectations (bounded rational and naïve) and studies both the steady state and dynamics for Cournot differentiated duopoly, showing explicit parametric conditions of the existence, local stability and bifurcation of the market equilibrium. Section 3 presents numerical simulations of the analytical findings, while also showing that complex behaviors through standard numerical tools (i.e., bifurcation diagrams, Lyapunov numbers, strange attractors and sensitive dependence on initial conditions). Section 4 concludes.

2. The Game

2.1. The construction of the game

We assume the existence of an economy with two types of agents: firms and consumers. There exists a duopolistic sector with two firms, firm 1 and firm 2, and every firm i produces differentiated goods whose price and quantity are given by pi and qi, respectively, with $i \in \{1, 2\}$. Their production decisions are taken at discrete-time periods t (t=0,1,2,...). In this study we consider heterogeneous players and more specifically, we consider that the firm 1 chooses the production quantity in a rational way, following an adjustment mechanism (bounded rational player), while the firm 2 decides by naïve way, selecting a quantity that maximizes its output (naïve player). The inverse demand function is given by the following equation:

$$p_i = a - q_i - dq_j, \quad \text{with } i \neq j$$

So, we have for each firm the following functions:

$$p_1 = a - q_1 - dq_2 \quad \text{and} \quad p_2 = a - q_2 - dq_1 \tag{1}$$

where α is a positive parameter which expresses the market's size and $d \in (-1, 1)$ is the parameter that reveals the differentiation degree between two products. It is understood that for positive values of the parameter d the larger the value, the less diversification we have between two products. If $d = 0$ each firm participates in a monopoly. On the other hand negative values of the parameter d are described that the two products are complementary.

We suppose the following cost functions:

$$C_i\left(q_i\right) = c \cdot q_i \tag{2}$$

where c > 0 is the marginal cost for two firms.

With these assumptions the profits of the firms are given by:

$$\mathrm{P}_i\left(q_i, q_j\right) = p_i q_i - C_i\left(q_i\right) = \left(\alpha - c - q_i - dq_j\right) q_i, \quad \text{with } i \neq j, \forall i, j = 1, 2 \tag{3}$$

Then the marginal profits at the point of the strategy space are given by:

$$\frac{\partial \mathrm{P}_i}{\partial q_i} = a - c - 2q_i - dq_j, \quad \text{with } i \neq j, \forall i, j = 1, 2 \tag{4}$$

We suppose that first firm is a bounded rational player. Then, if k > 0 the dynamical equation of the first player is:

$$\frac{q_1(t+1) - q_1(t)}{q_1(t)} = k\frac{\partial \mathrm{P}_1}{\partial q_1} \tag{5}$$

where k is the speed of adjustment of player 1, it is a positive parameter which gives the extent of production variation of the firm following a given profit signal. We assume that the second firm searches for information about firm 1 and after that it makes an estimation based on this information. We assume it is hard to get the perfect information about first player's exact decision, but firm 2 can get a bit of effective information about the exact decision. We use $\lambda \in [0, 1]$ to denote the amount of effective information, the largest value of l means a more accurate estimation. Considering the fact that second firm's basic information is first firm's last decision $q_1(t)$ and firm 2's perfect information should be the exact decision of this period $q_1(t+1)$, we assume the estimation depending on l takes the following form:

$$q_1^e(t+1) = (1-\lambda)\, q_1(t) + \lambda q_1(t+1) \tag{6}$$

where λ=0 means firm 2 only gets the basic information about firm 1, λ=1 means firm 2 gets the perfect information about firm 1. Based on this estimation, the second firm decides with naïve way by selecting a production that maximizes its profits (naïve player):

$$q_2(t+1) = \arg\max_{q_2} \mathrm{P}_2\left(q_1^e(t+1), q_2(t+1)\right) \tag{7}$$

The dynamical system of the players is described by:

$$\begin{cases} q_1(t+1) = q_1(t) + kq_1(t) \cdot \dfrac{\partial \mathrm{P}_1}{\partial q_1} \\ q_2(t+1) = \dfrac{a - c - dq_1(t) - d \cdot k \cdot \lambda \cdot q_1(t) \cdot \frac{\partial \mathrm{P}_1}{\partial q_1}}{2} \end{cases} \tag{8}$$

We investigate the effect of the information parameter λ on the dynamic of this system.

2.2. Dynamical analysis

2.2.1. The equilibriums of the game

The equilibriums of the dynamical system Eq.(8), which obtained by setting

$$q_1(t+1) = q_1(t) = q_1^*, q_2(t+1) = q_2(t) = q_2^*$$

in the system Eq. (8), are the nonnegative solutions of the algebraic system:

$$\begin{cases} k \cdot q_1^* \cdot \dfrac{\partial \mathrm{P}_1}{\partial q_1} = 0 \\ q_2^* = \dfrac{a - c - dq_1^* - d \cdot k \cdot \lambda \cdot q_1^* \cdot \frac{\partial \mathrm{P}_1}{\partial q_1}}{2} \end{cases} \tag{9}$$

- If $q_1^* = 0$, then $q_2^* = \frac{a-c}{2}$ and we obtain the boundary equilibrium:

$$E_0 = \left(0, \frac{a-c}{2}\right) \tag{10}$$

- If $\frac{\partial P_1}{\partial q_1} = \frac{\partial P_2}{\partial q_2} = 0$, then we form the following system:

$$\begin{cases} q_1^* = \dfrac{a - c - dq_2^*}{2} \\ q_2^* = \dfrac{a - c - dq_1^*}{2} \end{cases} \tag{11}$$

The solutions of this system are:

$$q_1^* = q_2^* = \frac{(a-c)(2-d)}{4-d^2},$$

and the Cournot-Nash equilibrium of the dynamical game is :

$$E_* = \left(q_1^*, q_2^*\right) = \left(\frac{(a-c)(2-d)}{4-d^2}, \frac{(a-c)(2-d)}{4-d^2}\right) \tag{12}$$

2.2.2. Stability of equilibriums

In order to study the local stability of equilibrium points of the model Eq.(8), we consider the Jacobian matrix $J\left(q_1, q_2\right)$ along the variable strategy $\left(q_1, q_2\right)$:

$$J\left(q_1, q_2\right) = \begin{bmatrix} f_{q_1} & f_{q_2} \\ g_{q_1} & g_{q_2} \end{bmatrix} \tag{13}$$

Where

$$\begin{aligned} f\left(q_1, q_2\right) &= q_1 + kq_1 \cdot \frac{\partial \mathrm{P}_1}{\partial q_1} = q_1 + k \cdot q_1\left[a - c - 2q_1 - dq_2\right] \\ g\left(q_1, q_2\right) &= \frac{a - c - dq_1 - d \cdot k \cdot \lambda \cdot q_1 \cdot \frac{\partial \mathrm{P}_1}{\partial q_1}}{2} \end{aligned} \tag{14}$$

Then

$$J\left(q_1, q_2\right) = \begin{bmatrix} 1 + k\left(\dfrac{\partial \mathrm{P}_1}{\partial q_1} + q_1^* \cdot \dfrac{\partial^2 \mathrm{P}_1}{\partial q_1^2}\right) & -dkq_1^* \\ -\dfrac{d}{2} - \dfrac{dk\lambda}{2}\left(\dfrac{\partial \mathrm{P}_1}{\partial q_1} - 2q_1^*\right) & \dfrac{d^2k\lambda q_1^*}{2} \end{bmatrix} \tag{15}$$

For the equilibrium E_0:

$$J\left(E_0\right) = \begin{bmatrix} 1 + k \cdot \dfrac{\partial \mathrm{P}_1}{\partial q_1} & 0 \\ -\dfrac{d}{2} - \dfrac{dk\lambda}{2} \cdot \dfrac{\partial \mathrm{P}_1}{\partial q_1} & 0 \end{bmatrix} \tag{16}$$

with trace

$$Tr\left[J\left(E_0\right)\right] = 1 + k\left(a - c - dq_2^*\right) = 1 + k \cdot \frac{(a - c)(2 - d)}{2}$$

and determinant

$$Det\left[J\left(E_0\right)\right] = 0.$$

The characteristic equation of $J\left(E_0\right)$ is:

$$x^2 - Tr \cdot x + Det = 0 \tag{17}$$

The solutions of Eq. (17) are the following eigenvalues of the Jacobian matrix:

$$x_1 = 0 \quad \text{and} \quad x_2 = 1 + k \cdot \frac{(a-c)(2-d)}{2} \tag{18}$$

Since $a - c > 0$, it's clearly that $|x_2| > 1$, and the equilibrium E_0 is unstable.

We now study the local stability properties of the Cournot– Nash equilibrium Eq. (12). In this equilibrium E_* the Jacobian matrix is:

$$J\left(E_*\right)\begin{bmatrix} 1-2kq_1^* & -kdq_1^* \\ -\frac{d}{2}+dk\lambda q_1^* & \frac{d^2k\lambda q_1^*}{2} \end{bmatrix} \tag{19}$$

with

$$Tr\left[J\left(E_*\right)\right] = 1 - 2kq_1^* + \frac{d^2k\lambda q_1^*}{2} \tag{20}$$

and

$$Det\left[J\left(E_*\right)\right] = \frac{d^2kq_1^*}{2}(\lambda - 1) \tag{21}$$

The stability conditions for a system in two dimensions with discrete time are generically given by (see, e.g., Elaydi S., [11], Gandolfo G., [13]):

$$\begin{aligned} &(i)\ \mathrm{H} = 1 - Det > 0 \\ &(ii)\ \mathrm{TC} = 1 - Tr + Det > 0 \\ &(iii)\ \mathrm{F} = 1 + Tr + Det > 0 \end{aligned} \tag{22}$$

The violation of any single inequality in Eq. (22), with the other two being simultaneously fulfilled leads to:

(i) a flip bifurcation (a real eigenvalue that passes through –1) when F=0

(ii) a fold or transcritical bifurcation (a real eigenvalue that passes through +1) when TC=0

(iii) a Neimark–Sacker bifurcation (i.e., the modulus of a complex eigenvalue pair that passes through 1) when H=0,

Since

$$1 - Det = 1 + \frac{d^2k(1-\lambda)}{2} \cdot q_1^* > 0, \tag{23}$$

and

$$1 - Tr + Det = kq_1^* \cdot \frac{4 - d^2}{2} > 0 \tag{24}$$

The conditions (i) and (ii) of Eq. (22) are always fulfilled. Then the condition (iii) is the condition for the local stability of the Nash equilibrium.

Since

$$1 + Tr + Det > 0 \Leftrightarrow \frac{1}{2} + \frac{2}{d^2} - \frac{2}{d^2 kq_1^*} < \lambda \leq 1 \tag{25}$$

we obtain

Proposition: *The Nash equilibrium of the dynamical system Eq.(8) is locally asymptotically stable if:*

$$\frac{1}{2} + \frac{2}{d^2} - \frac{2}{d^2 kq_1^*} < \lambda \leq 1 \tag{26}$$

where

$$q_1^* = \frac{(a - c)(2 - d)}{4 - d^2} \quad \text{and} \quad d \in (-1, 1), k \in (0, 1)$$

3. Numerical Simulations

The main purpose of this section is to show that the qualitative behavior of the solutions of the duopoly game with heterogeneous player described by the dynamic system Eq. (8), can generate, in addition to the local flip bifurcation and the resulting emergence of a two-period cycle, chaotic behavior.

To provide some numerical evidence for the chaotic behavior of the system Eq.(8), as a consequence of change in the parameter λ of the information degree, we present various numerical results, including bifurcations diagrams, strange attractors, Lyapunov numbers and sensitive dependence on initial conditions (Kulenovic, M. and Merino, O. [16]). For this, it is convenient to choose the following parameter set only for illustrative purposes$a = 6.5$, $c = 0.5$, $k =$ 0.61, $d = 0.94$. In this case the stability condition becomes:

$$0.944 < \lambda \leq 1 \tag{27}$$

Numerical experiments are computed to show the bifurcation diagram with respect to λ, strange attractors of the system Eq. (8) in the phase plane (q_1, q_2) and Lyapunov numbers. Figure 1 shows the region of stability between λ and k (left) or d (right). Figure 2 shows bifurcation diagrams with respect to the parameter λ. The figure clearly shows that a decrease in the extent of information parameter λ (i.e., the parameter λ moves from 1 to values smaller than 1), implies that the map Eq. (8) converges to a fixed point for 0.944 < λ < 1. Starting from this interval, in which the Cournot- Nash equilibrium is stable. Fig. 2 shows, also, that the equilibrium undergoes a flip bifurcation at λ=0.944. Then, a further decrease in information parameter implies that a stable two-period cycle emerges for 0.35 < λ < 0.944. As long as the parameter λ reduces a four-period cycle, cycles of highly periodicity and a cascade of flip bifurcations that ultimately lead to unpredictable (chaotic) motions are observed when λ is very small. As an example, the phase portrait of Fig. 4 depicts the strange attractor for λ= 0.04. This is the graph of the orbit of (0.1, 0.1) with 2000 iterations of the map Eq. (8) for $a = 6.5$, $c = 0.5$, $k =$ 0.61, $d = 0.94$ and $\lambda = 0.04$.

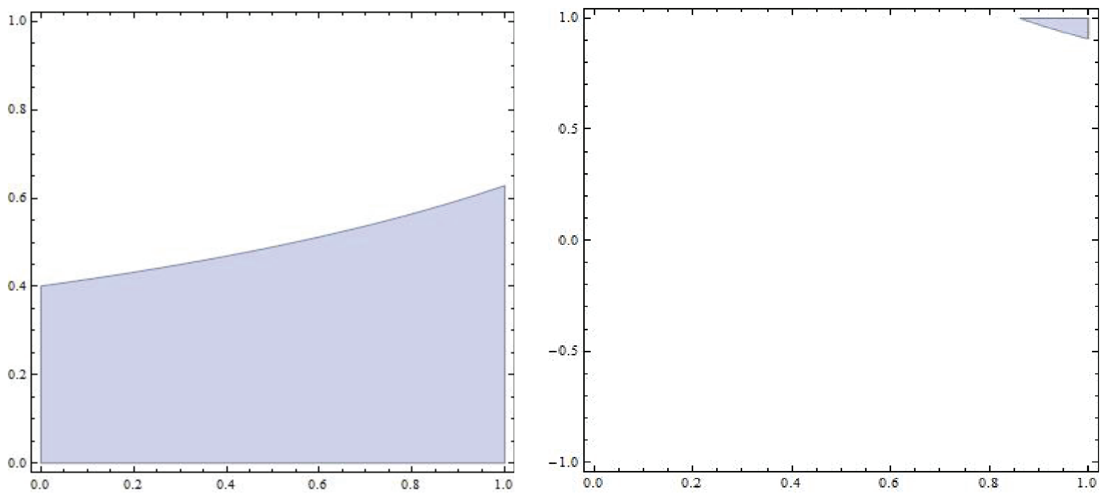

Figure 1: Region of stability between λ (horizontal axis) and k (left) or d (right) (vertical axis) of Eq. (26) for (left) and for $a = 6.5, c = 0.5, k = 0.61$ (right).

Another numerical tool useful in order to determine the constellation of parameters for which trajectories converge to periodic cycles, quasi-periodic and chaotic attractors, is the study of the Lyapunov number (i.e. the natural logarithm of Lyapunov exponent) as a function of the parameter of interest (which, in the present paper, is assumed to be the degree of information λ). As is known, there exists evidence for quasi periodic behavior (chaos) when the Lyapunov number is equal to one (greater than 1). Figure 4 shows Lyapunov numbers of the same orbit. If the Lyapunov number is greater of 1, one has evidence for chaos.

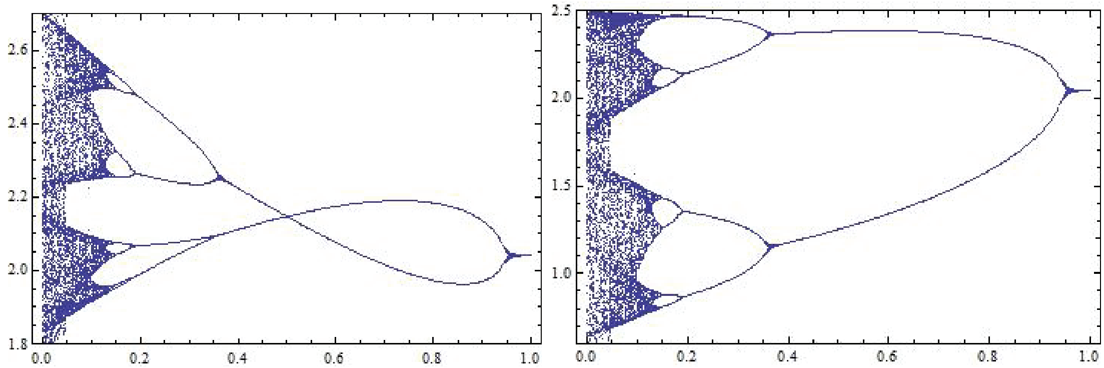

Figure 2: Bifurcation diagrams with respect to the parameter λ against variable (left) and (right), with 400 iterations of the map Eq. (8) for $a = 6.5, c = 0.5, k = 0.61, d = 0.94$.

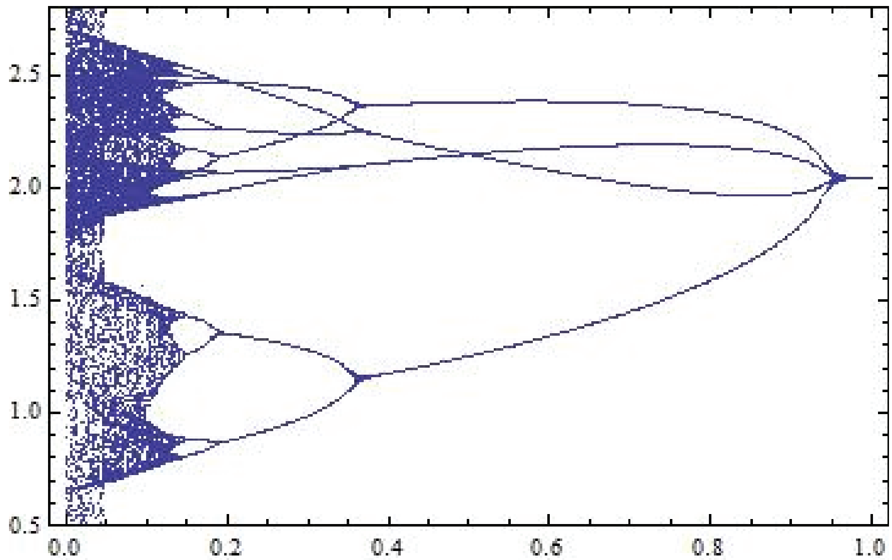

Figure 3: Two bifurcation diagrams of Fig.2 are plotted in one.

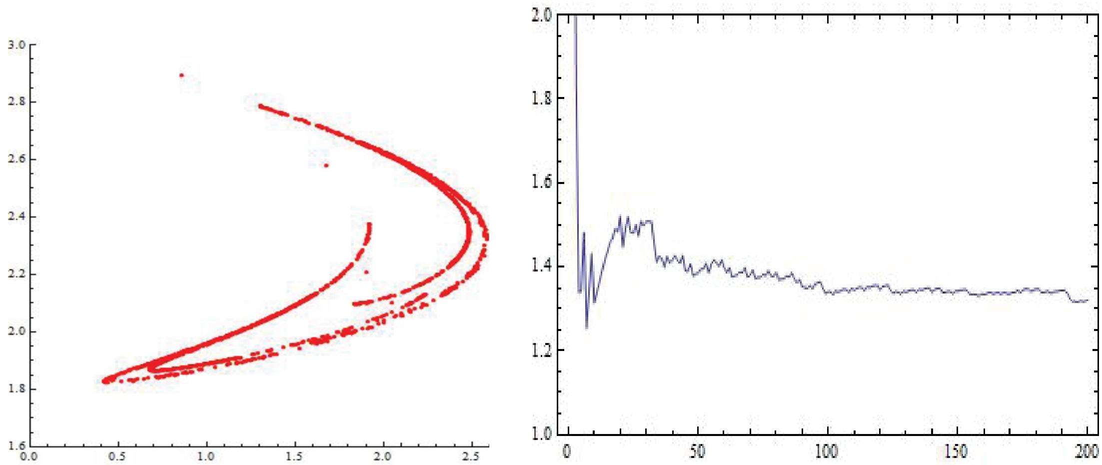

Figure 4: : Phase portrait (strange attractors) and Lyapunov numbers of the orbit of (0.1,0.1) with 2000 iterations of the map Eq.(8) for $a = 6.5, c = 0.5, k = 0.61, d = 0.94$ and for $\lambda = 0.04$.

As is known, the sensitivity dependence to initial conditions is a characteristic of deterministic chaos. In order to show the sensitivity dependence to initial conditions of system Eq. (8), we have computed two orbits with initial points (0.1, 0.1) and (0.101, 0.1), respectively. Figure 5 shows sensitive dependence on initial conditions for q_1-coordinate of the two orbits, for the system Eq. (8), plotted against the time with the parameter values $a = 6.5$, $c = 0.5$, $k = 0.61$, $d = 0.94$ and $\lambda = 0.04$. At the beginning the time series are indistinguishable; but after a number of iterations, the difference between them builds up rapidly. From these results when all parameters are fixed and only λ is varied the structure of the game becomes complicated through period doubling bifurcations, more complex bounded attractors are created which are aperiodic cycles of higher order or chaotic attractors.

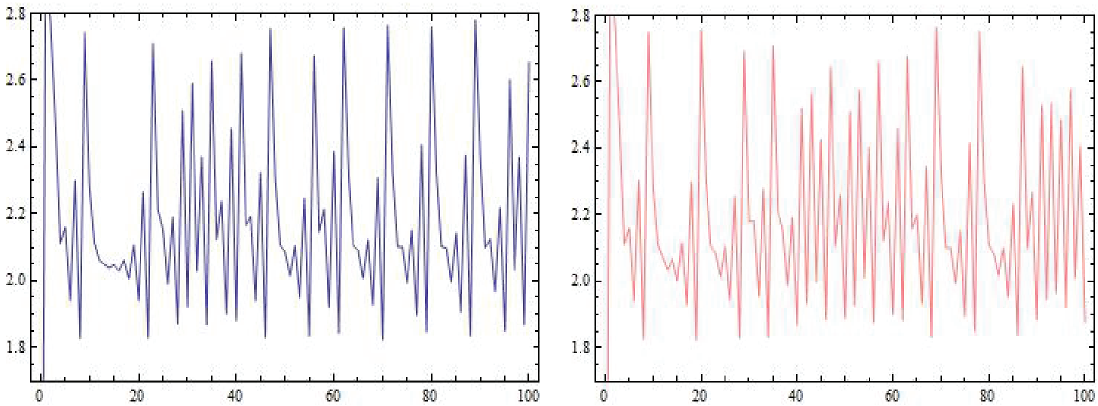

Figure 5: Sensitive dependence on initial conditions for q1-coordinate plotted against the time: the two orbits: the orbit of (0.01, 0.01) (left) and the orbit of (0.0101, 0.0101) (right), for the system Eq. (8), with the parameters values for $a = 6.5, c = 0.5, k = 0.61, d = 0.94$ and for $\lambda = 0.04$.

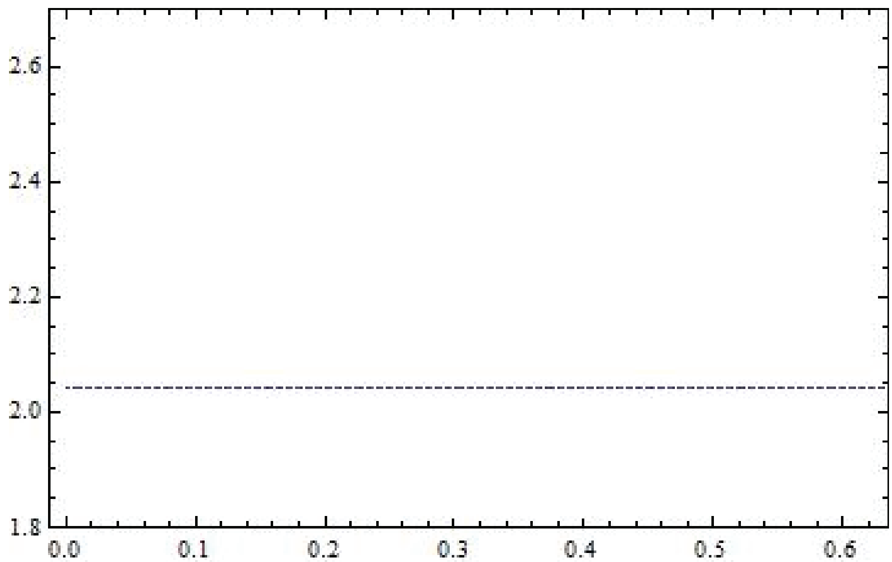

Figure 6: Bifurcation diagrams with respect to the parameter λ against variable q_1 and q_2, with 400 iterations of the map Eq. (8) for $a = 6.5$, $c = 0.5$, $k = 0.4$, $d = 0.94$.

But if we choose $a = 6.5$, $c = 0.5$, $k = 0.4$, $d = 0.94$ the stability condition becomes:

$$\lambda > -0.014 \tag{28}$$

From Eq. (28) for each λ in the interval [0, 1] the Nash equilibrium is locally asymptotically stable (Figure 6). Therefore, for these values there are stable trajectories and a higher or lower degree of information does not destabilize the market.

4. Conclusions

In this paper we analyzed the dynamics of a differentiated Cournot duopoly with heterogeneous expectations, linear demand and cost functions. One player searches for information about his rival before his decision and we investigated the effects of the information in this dynamic game. The main result is that a lower degree of information may destabilize the unique Cournot–Nash equilibrium. While also showing the existence of deterministic chaos. But, we showed also that, if the products are almost homogeneous (high value of the differentiation parameter) and for lower values of the speed of adjustment, there are also stable trajectories for each value of the level of information. The economic intuition behind the result is that if the degree of product differentiation is small (i.e. fiercer competition), the information parameter does not always destabilize the market.

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