KnE Social Sciences

ICMScE
International Conference On Mathematics And Science Education
Volume 2024

Research Article

Analysis of Students' Combinatorial Thinking Model in Solving Combinatorics Problems

Rita Desfitri

Universitas Bung Hatta, Jalan Sumatera Ulakkarang, Padang 25133, Indonesia

ORCID

Rita Desfitri: https://orcid.org/0000-0001-6450-8733

Abstract.
The purpose of this study was to analyze students' combinatorial thinking in solving combinatorics problems. This study was organized into two related areas. The first focused on whether students applied combinatorial thinking in Lockwoods' model, and the second emphasized on students' ability in solving the given combinatorics problems. This research method used was a qualitative descriptive method. Participants of this study were students majoring mathematics education at Bung Hatta University who took combinatorics courses. Five combinatorics problems were given to the students and then analyzed. The result showed that based on Lockwoods' model, the aspect of combinatorial thinking that was more widely applied in solving combinatorics problems was the counting process. However, most students who were able to solve problems quite well, and tended to solve problems with steps from formulating, followed by the process of counting to reach a set outcome. Whereas students who did the counting process stage without formulating problems into mathematics expressions, generally checked a set of outcomes by trial and errors. It can also be noted that in general, f/or any problem given, the percentage of students who faced difficulties when formulating problems into mathematical expressions or on the counting process was more that 50%, and the number of students who reached the correct set of outcomes was relatively low.

Keywords: student, combinatorial thinking model, solving combinatorics problems

Corresponding Author: Rita Desfitri; email: Rita Desfitri: https://orcid.org/0000-0001-6450-8733

Published: 26 April 2024

Publishing services provided by Knowledge E

Selection and Peer-review under the responsibility of the ICMScE Conference Committee.

1. INTRODUCTION

As a branch of discrete mathematics, combinatorics has become more well-known in our society. Combinatorics is one of branches in discrete mathematics concerning the study of finite or countable discrete objects which related to many area of mathematics such as number theory, probability theory, coding, graph and group theory. Basic combinatorial concepts and enumeration have been discussed since ancient world. Combinatorics could be classified into many different ways. If it was lenses from framework of counting, combinatorics could be looked up into different lenses: enumerative and analytic combinatorics. Enumerative combinatorics was more classical which focused on counting numbers or certain object used explicit formula which also deals with real life problems using the basic principle of counting. Analytic combinatorics deals with enumeration of combinatorial structures from mainly analysis and probability theory.

Furthermore, as a part of discrete mathematics and its relevant to modern life, it was one of reasons why combinatorics become important for mathematics education students. The National Council of Teachers of Mathematics even raised the issue of teaching of discrete mathematics topics in mathematics education. Three important areas of discrete mathematics are integrated within these standards: combinatorics, iteration and recursion, and vertex-edge graphs [1]. Combinatorics is an important of mathematics and mathematics education since students need to know and understand combinatorics to solve real life problems. In addition, in recent years, teachers of mathematics, technology and science in schools have stated that discrete mathematics is closely related to fields of study such as computer sciences, statistical opportunity theory and business management which are very much needed in modern life [2].

Many researches have been conducted in combinatorics. Gadino in [3] for example, analyzed how students answered problems related to combinatorics-based solving of semiotic. Melusova, and Vidermanova in [4] also examined how combinatorics poblems solving formulated, before and after strategy given. There were some fundamental concepts that took major roles in understanding and solving combinatorial problems. In real life, there are so many combinatorial problems that need to be solved. For example: if there are only two bedrooms for five guests, how many persons will stay in each room? Of course it is impossible to say that each room will be occupied by two and half persons. So, in this case, combinatorial problems play an important role in such problems. The basic concept of permutation, combination, are also main discussion to handle everyday problems as combinatorial problems so that they can be solved in a proper pattern.

Combinatorial problems, could in fact, be developed in many different level of students, even from elementary up to university, depending on their complexity of thinking. Combinatorial problems involve finding the grouping, sorting or ordering, assignment of a finite and discrete set of objects that satisfy particular conditions. Grouping, ordering, and sorting could be the simplest combinatorial problems that can be applied to elementary level. Counting problems are also kind of combinatorial problems which fosters deep mathematical thinking but are also source of students at a variety of level [5, 6]. Counting problems which are usually easy to state but difficult to solve indicate that solving combinatorial problems needs deep mathematical thinking process. Combinatorial problems stimulate students' way of thinking in constructing meaningful problem presentation, provide logical reasons, and are able to generalize concepts in mathematics [7]. There are many kind of combinatorial problems. Batanero in [6] also classified combinatorial problems into three different categories: existence problems deal with whether solutions does exist, counting problems which investigate how many solutions may exist; and optimization which focused on finding a best solution (if any) for a given particular problems.

Although concepts such as permutation and combination have been taught in high school, in combinatorics courses at university level, many number of students had some difficulties in mastering lecture and solving problems given during lectures. From our experience and perspectives on combinatorial teaching and learning, as well as many discussions around the world, there was much evident literally noted that some students struggle with solving combinatorial problems. Learning combinatorial concepts require a special ways of thinking that some researchers acknowledge it [8] Combinatorial thinking can be viewed as a process of finding a number of alternative solution of discrete problems. Regarded combinatorial thinking as a tool to find a systematics way so that all possibilities included. Combinatorial thinking is also an essential element in comparison with other type of logical thinking and its existence is an important mathematical learning [1].

One way that can help us see how students solve combinatorial problems is by referring to their combinatorial thinking. Lockwood in [9] introduced a combinatorial thinking model that explains coordination of sets and processes in combinatorial problem solving. In this model, students' combinatorial thinking consist of three components, namely mathematical formula or expression, counting process, and set of outcomes or a series of answers. A formula or formula component refers to an evaluable mathematical expression that is often thought of as an answer to calculation problems. The counting process refers to the actual steps that in person is physically and mentally involved in doing the calculation. Meanwhile, a series of answers to a given problems refers to desired result from solving problems. For more detail, students' combinatorial thinking model can be described in Figure 1.

According to Lockwood in [10], the relationship between counting processes and formulas/expression is not trivial and become an important part of understanding what is involved in solving counting problems. It reflect that a given formula/expression may elicit a counting process. It mean that a given mathematics formula can be naturally associated with a counting process. From opposite direction, it is possible to conceptualize counting process that generate an appropriate formula [9]. The relationship between the counting process and set of outcomes indicate that a counting process can be seen as generating or organizing some set of solution, and in contrary, it is possible to

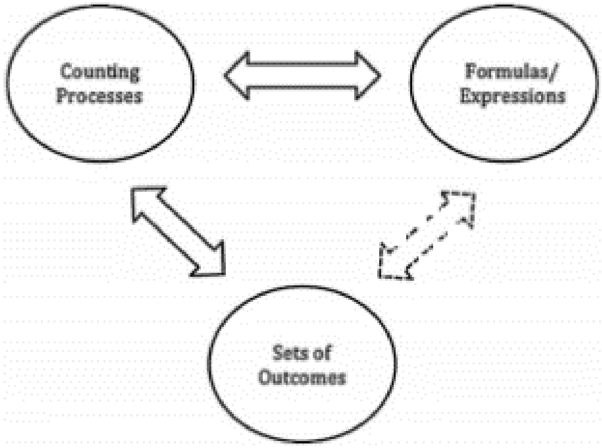

Figure 1: A model of combinatorial thinking by Lockwood.

arrive at a counting process from a set of outcomes, where which mean that by starting with set of outcomes, and then organize the set in particular way. Lockwood in [9] also explained that the relationship between formula/expression and set of outcomes was dotted because it was less clearly linked than the other two.

At Bung Hatta University, combinatorics is one of courses offered to 7th semester students in mathematics education department. The lessons discussed are continuation of what has been learned in high school, but are more in depth. This concerns the basic principle of counting, permutation and combination, optimization and the shortest route, as well as binomial and multinomial concepts. How ever, the data shows that students do not master the lessons taught during lectures, such as permutations and combinations that have even been discussed in high school. This can also be seen from the combinatorics course scores which tend to be low because the majority get C or D grades as shown in the Table 1.

Table 1: Distribution of combinatorics course scores for students of the Bung Hatta University Mathematics Education study program in the last two academic years.

Academic Year	**Grade**				**Number of students**
	A	**B**	**C**	**D**	
2017/2018	11	21	30	4	66
2018/2019	1	9	19	6	35

Source: documentation of combinatorics course grades on Bung Hatta University portal

Similar to data on Table 1, Lockwood [10] noted that there is much documented evidence for the fact that students struggle with solving combinatoial problems correctly. In order to check students' thinking in combinatoial problems, Lockwood propose a model of combinatorial thinking that explore how students approaches to combinatorial problem solving, highlighting relationships between counting formula/expression, counting process, and set of outcomes. Based on the data above, it is necessary to conduct research to see how students at the Department of Mathematics Education, Bung Hatta University solve combinatorics problems, regarding their combinatorial thinking. The objectives of this study were to figure out and describe the followings: 1) Students performed their combinatorial thinking based on Lockwoods' model; 2) Students' ability in formulating and doing counting process when solving combinatorics problems. The students' combinatorial thinking model in this study was figured out by looking at the relationship among components of Lockwood' model applied on students' answer. Relationship between components were investigated based on the following guidelines:

Table 2: List of questions or combinatorial problems.

Relationship between components	**Directions**	**Indicators**
Counting process and mathematics expressions (formula)	mathematics expressions → counting process	Students attribute combinatorial meaning to mathematics expressions in the form of enumeration process
	counting process → mathematics expressions	Students conceptualize a counting process that generates mathematics formula
Counting process and set of outcomes	counting process → set of outcomes	Students apply counting process to generate some set of outcomes
	Set of outcomes → counting process	Students consider some possible solutions and breakdown the outcomes that arrive at a counting process

2. RESEARCH method

This study was descriptive research to figure out how students learned and solved combinatorial problems. Participants of this study were 20 students who enrolled in combinatorics course on their 7^{th} semester at mathematics department, University of Bung Hatta. The data collected during the academic year 2019/2020. In order to ensure the instrument content validity, the instrument used was selected from mid and end semester exams constructed by lecturer teaching combinatorics course which referred the course syllabus. There were eleven questions from both mid-term and end semester exams which consisted of five and six questions respectively, but the researcher only picked up five questions to be used for this research.

These five problems were related to the basic principle of counting; the principle of inclusion-exclusion; permutation and combination; and the shortest route problems. For every given problem, students' answers were analyzed by assessing the following: 1) Their complexity of three component of Lockwood combinatorial thinking model; 2) In what direction they solved problems based on the relationships between components according to Lockwood' model of combinatorial thinking; 3) Students' achievement level by looking up their score in solving problem given. The combinatorial problems given can be seen in Table 3

Table 3: List of questions or combinatorics problems.

Problem number	Topics	Score	Problems
#1	Basic Principle of addition	20	How many of pair integers (a, b) such that sum of the squares of these integers is less than or equal to 7?
#2	Permutation	20	How many three-digit non-negative integers do not contain 5?
#3	Combination without repetition	20	There are six guests who will be seated around a round table. If there are two of them who are not allowed to sit next to each other, how many possible seating arrangements for the six guests?
#4	Combination with repetition	20	Five students go to campus restaurant for breakfast. There are three different types of food offered there, namely fried rice, fried noodles, and chicken soup. If each student orders food, and can only order exactly one type of food, how many different order combinations do the five students have?
#5	Combination with inclusion exclusion	20	Four cities, namely cities A, B, C, and D are located sequentially on the Cartesian plane where $A(0,0)$, $B(8,6)$, $C(4,3)$, and $D(5,3)$. If a person wants to travel from city A to city B, but cannot take the route connecting city C and city D, how many shortest routes can he choose?

3. result and discussion

The result of this study was organized into two major sections. The first was focused on whether students applied combinatorial thinking in terms of Lockwood's model, and the second was emphasized on students' ability in solving given problems. From data obtained based on students' work sheets, number of students who applied combinatorial thinking aspect in terms of relationship between its components can be looked at in Table 4.

Table 4: Number of students who formulate problems into mathematical expressions and apply counting process in either direction.

Relationship between components	Directions	Number of students who work on problems				
		#1	**#2**	**#3**	**#4**	**#5**
Counting process and mathematics expressions (formula)	mathematics expressions → counting process	10	10	10	7	10
	counting process → mathematics expressions	6	4	5	6	5
Counting process and set of outcomes (solutions)	counting process → set of outcomes	10	10	15	13	15
	Set of outcomes → counting process	10	10	5	7	5

Table 4 showed us that in all problems, regarding the relationship between mathematics formulas and counting process, the majority of the students worked in the direction of formulating math formulas followed by counting process. As for the relationship between counting process and set of outcomes, students perform in the direction of counting process to reach some of outcomes. One interesting is that for problem #1 and #2, where half of them work on counting process to set of outcomes directions and another half of them on the reverse direction. This possible since problem #1 and problem #2 were simple questions so it was possible for students to start solving problems by trying the series of possible answer that might match or meet the constrains given.

For problem #1, some students performed counting process by trying to pick up some possible answers considering the constraint of given solutions and finally found out the whole possible outcomes/solutions. However, for questions #2, students who were able to carry out counting process are only those who have formulated problems into mathematical formula. Although the process of considering three digits for their positions indicating hundreds, teens, and unit written sequentially hold an important rule for performing the counting process. It would be awkward to find all three-digit numbers and choose which numbers do not have 5s on them manually without one being left behind. If we look at whether students apply aspect of combinatorial thinking in correct way, then data can be seen as in Table 5.

Table 5: Number of students who formulate problems into mathematical expressions and apply counting process in either directions correctly.

Relationship between components	**Directions**	**Number of students who work on problems correctly**				
		#1	**#2**	**#3**	**#4**	**#5**
Counting process and mathematics expressions (formula)	mathematics expressions → counting process	3	5	5	7	5
	counting process → mathematics expressions	3	2	3	2	4
Counting process and set of outcomes (solutions)	counting process → set of outcomes	5	5	8	8	5
	Set of outcomes → counting process	2	1	0	0	0

From data, it gave pictures about a tendency for all students to perform counting process, but not all begin with formulating formula into mathematics expressions. Based on their students' answer, their reasons varied, they had no idea how to formulate problems into math expression, and other said that they just started by trying with the series of answers that might match or meet the given condition. It means that for some cases with limited number of possible solutions, it is possible for students to immediately carry out counting or calculation process without formulating problems into math expression/formula (problems #1, #2, and #3). There were also some interesting things can be seen from Table 5 as follows: First, students could successfully work on set of outcomes to counting process directions only for problems #1 and #2. All of them who did it were those who did not pay attention on math formula but tried to go directly to all possible solutions (by trials and errors) and did some counting process. The process of trial and errors (although not always successful) since for problem #1 and #2 all possible solutions are quite limited.

Second, on the contrary or problems #3, #4, and #5 where there existed the large number of solutions, then finding or counting solution by trial and errors is not desired. In this case the math formulas were needed, so none of those who worked on direction from set of outcomes to counting process, and at the same time they did not pay attention on formulating problems into math formula, successfully solved problems. For problem #3, for example, students with good understanding on the given question realize that it was a circular permutation with 6 elements, so they were already aware that for n elements, the total linear permutation of n is $P = n!$ However, only eight students out of fifteen students trying to formulate problems were aware for circular permutation where n linear permutations could be assumed as 1 circular permutation. It leaded to the original formula for circular permutation as

$$\frac{n!}{n} = (n-1)!\,(1)$$

Similar to this, since problem #4 needed to aware of many different conditions, then doing counting process straight forward to generate an appropriate formula might be a little difficult. They also ignored (or forgot) about the certain food that might not be ordered by anyone, or in contrast, everyone ordered the same food. Some students who tried to formulate problem were successfully formulating mathematics expression correctly and interpret problem as *Diophantine case*. With the formula that was formed in advance, crucial things for counting process were also covered immediately. They tried to form the equation

$$a + b + c = 5(2)$$

Where non-negative integers a, b, and c, showed the number of students who ordered fried rice, fried noddle, and chicken soup respectively.

For question #5. Students who tried to formulate problem from counting process they perform did not reach correct formula, since they were also unable to express constrains in the problems or required condition. An interesting thing was that five students tried to work on possible set of solutions before counting process or formulating formula, but then they all failed to do so. In case of the shortest route problem with some requirements, it was difficult to find all possible solution without formulating problems and appropriate counting process, because the total of possible solutions is enormous. From data and discussion above, if Lockwood's combinatorial thinking introduced two direction relationship between mathematics formulae and counting process, as well as relationship between counting process and set of solutions, it can be shown that majority of the students solved problems in the following steps as in Figure 2.

Figure 2: Trends of students' thinking model to solve combinatorial problems.

From questions given, students' answers were checked and graded. Each question had a maximum score of 20 points, and distribution of scores and grades obtained by 20 students as research participants can be seen in the Table 6.

TABLE 6: Distribution of score and graded of students.

Score Range	Grade	Numbers of students	Percentage (out of 20)
81 - 100	A	0	0
65 - 80	B	6	30
55 - 64	C	8	40
45 - 55	D	6	30

From an in-depth look, in can be seen that Table 5 along with Table 6 showed us that students' ability in solving combinatorial problems was relatively low. Table 5 also showed that in general, for each given problem, less than half of the students were able to solve it correctly. For problem #3 and #4, only eight of students who reached the correct answer, even though it was still more than the number of students who answered correctly on other questions. It showed us more data that the percentage of students who are able to solve combination problems is higher.

For problem #5 (about the shortest route from city A to city B), it was the least number of students who were able to find the expected solutions. Although there were fifteen students who tried to visualize the problems (regardless of it based on the direction of math expression to counting process or vice versa), they noted that the total steps needed was only fourteen. They were aware of the formula that once someone moved either left of down direction, then the route could no longer considered the shortest as shown in the Figure 3.

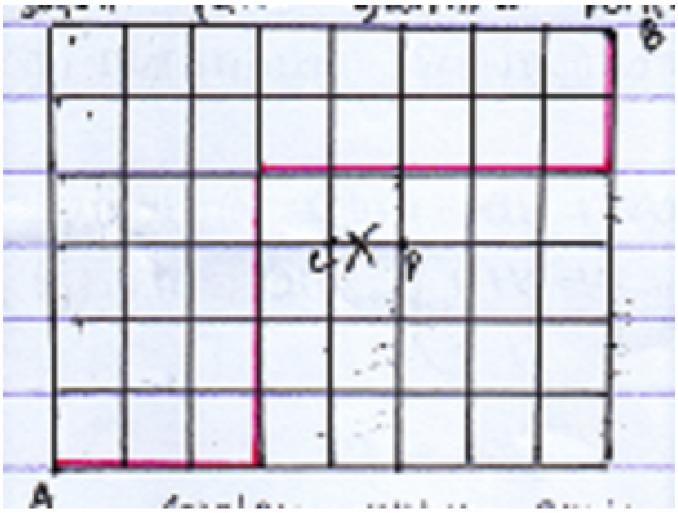

Figure 3: Example of students' visualization for problem #5.

Unfortunately, from nine students who were able to visualize the problem, only five of them were aware of the constraint that need to be considered, which was not being allowed to cross the path connecting city B and city C. It told us that had difficulties and were unable to complete the answer as expected if the problems had several constraints. Besides some students were unable to define math expression for given problems, this also happened because some students failed to carry out counting process in appropriate way even though he/she already got a correct formula.

Many students have some difficulties in understanding the concepts and in solving problems given. Many students did not understand combinatorial problems given or have no idea what to do with calculation. Some students were also unable to find a series of possible answer to that problem. As Crespo in [11] figured out that problems which are easy to solve, could be kind of forms where hints are provided to lead someone to as solutions or answer, but student's ability in mastering lecture and solving combinatorial is a challenge because as the prospective teachers, this knowledge will help them later in teaching their own students [12].

4. CONCLUSION

From data and discussion, we figured out that trends of students who successfully solved problems, was to perform from formulating data (mathematics expression) followed by counting process to find a set of solutions. There was also a tendency that all students perform counting process, although not always preceded by defining mathematical formula. However, for problems involving a very simple counting process and a very limited set of solution, then it was possible to solve problems in the reverse direction by performing steps as by looking at possible solutions candidates to perform multiple computations. From this study, by looking at the students' ability in solving combinatorics problems and their combinatorial thinking, we figured out that although not all students formulate problems into math expression or formula, all students carried out combinatorial counting process thinking aspect. Students who performed counting process stage without formulating a mathematics expression, generally examine possible solutions by a trial-and-error approach.

Overall, it could be noted that not all students were able to formulate problems into math formula/expression, and the number of students who reached the correct set of solution was relatively low. This fact needed to be a concern so that in the future combinatorics teaching and learning could be improved. So, this study might contribute to our understanding of students' combinatorial thinking when solving combinatorics problems, and in particular, what difficulties students faced in combinatorial thinking.

References

[1] NCTM. National council of teachers of mathematics. Principles and standards for school mathematics. Reston (VA): NCTM; 2000.

[2] Pittman LC, Herman GL. "Documenting students' faulty schema and misconceptions about combinations and permutations," In: *2015 IEEE Frontiers in Education Conference (FIE)*. pp. 1–9. *IEEE* (2015). https://doi.org/10.1109/FIE.2015.7344035.

[3] Godino JD, Batanero C, Font V. The onto-semiotic approach to research in mathematics education. ZDM Math Educ. 2007;39(1–2):127–35.

[4] Melušová J, Šunderlík J. "Pre-service teachers' problem posing in combinatorics," In: et al Ondrej Sedivy, Ed. *The 12th Conference Proceeding. Faculty of Natural Science, Constantine of Philosopher, University of Nitra*. pp. 115–122. *University of Nitra* (2014).

[5] Mashiach Eizenberg M, Zaslavsky O. Students' verification strategies for combinatorial problems. Math Think Learn. 2004;6(1):15–36.

[6] Batanero C, Navarro-Pelayo V, Godino JD. Effect of the implicit combinatorial model on combinatorial reasoning in secondary school pupils. Educ Stud Math. 1997;32(2):181–99.

[7] Sriraman B, English LD. Combinatorial mathematics: research into practice. Math Teach. 2004;98(3):182–91.

[8] Rezaie M, Gooya Z. What do I mean by combinatorial thinking? Procedia Soc Behav Sci. 2011;11:122–6.

[9] Lockwood E. A model of students' combinatorial thinking. J Math Behav. 2013;32(2):251–65.

[10] Lockwood E. "Students approaches to combinatorial enumeration: The role of set-oriented htinking," http://archives.pdx.edu/ds/psu/7054, (2011).

[11] Crespo S. Learning to pose mathematical problems: exploring changes in preservice teachers' practices. Educ Stud Math. 2003;52(3):243–70.

[12] Desfitri R. Pre-service teachers' challenges in presenting mathematical problems. J Phys Conf Ser. 2018;948(1):012035.