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Parameter Estimation and Hypothesis Testing on Bivariate Log-Normal Regression Models

Kadek Budinirmala, Purhadi*, and Achmad Choiruddin

Department of Statistics, Faculty of Science and Data Analysis, Institut Teknologi Sepuluh November, East Java, Indonesia

ORCID
https://orcid.org/0000-0002-1592-6645
Achmad Choiruddin: https://orcid.org/0000-0003-2568-2274

Abstract.
This study aims to introduce a bivariate Log-Normal regression model and to develop a technique for parameter estimation and hypothesis testing. We term the model Bivariate Log-Normal Regression (BLNR). The estimation procedure is conducted by the standard Maximum Likelihood Estimation (MLE) employing the Newton-Raphson method. To perform hypothesis testing, we adapt the Maximum Likelihood Ratio Test (MLRT) for simultaneous testing with test statistics which, for large n, follows Chi-Square distribution with degrees of freedom p. In addition, the partial testing is derived from a central limit theorem which results in a Z-test statistic.

Keywords: parameter estimation, hypothesis testing, bivariate log, normal regression

Corresponding Author: Purhadi; email: purhadi@statistika.its.ac.id

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1. INTRODUCTION

Bivariate regression is a common approach when two correlated response variables are modeled simultaneously. For example, bivariate regression was used to predicting an approximate value of the AoR in the free-fowing and cohesive powders [1], and to estimating the odds of mortality due to Covid-19 and infection by Covid-19 [2]. When model the univariate regression, we only focus on mean and variance. However, for two variances, we should include the correlation [3]. For parameter estimation, Ordinary Least Square (OLS) is the standard approach since it is intuitively appealing, mathematically much simple, and closed form [4]. Although OLS is an appropriate approach for normally distribution data, this technique cannot work for other distribution. For example, the Three Parameter Bivariate Gamma Regression applied to cases of the Under-five Mortality Rate (U5MR) and the Maternal Mortality Rate (MMR) in North Sulawesi, Gorontalo, Central Sulawesi Provinces in 2016 [5], the Bivariate Bernoulli Regression for analyzing malnutrition data in Bangladesh [6], the Bivariate Poisson

Regression applied to estimating the change in soccer's home advantage during the Covid-19 pandemic in 13 European countries [7], and the Bivariate Poisson Inverse Gaussian Regression applied to Infant and Maternal Death Case Study in East Java 2017 [8].

In this study, we develop the Bivariate Log-Normal Regression (BLNR) model that can be used to modeling cases with two continuous response variables that correlate with each other, have positive skewness and heteroscedasticity [9]. For example, in the case of poverty we could observe the correlated rensponse, i.e, poverty gap index and poverty severity index. This issue cannot be handle by univariate version of Log-Normal Regression (LNR). This study can be regard as an extension of univariate LNR previously developed by [10, 11].

To build the BLNR model, we develop technique for parameter estimation and hypothesis testing of the BLNR model. For the parameter estimation we used the MLE method, if the results are not closed form and cannot be solved analytically, then continued with Newton-Raphson iterations. To perform hypothesis testing, for simultaneous testing uses the Maximum Likelihood Ratio Test (MLRT) method with test statistics which, for large n, follows chi-square distribution with degrees of freedom p. For the partial testing derived from a central limit theorem which results in a Z-test statistic.

2. RESEARCH METHOD

2.1. Log-Normal Regression (LNR)

Suppose Y has a Log-Normal distribution with parameters μ_Z and σ_Z^2, denoted by $Y \sim LN\left(\mu_Z, \sigma_Z^2\right)$, where μ_Z is the scale parameter and σ_Z^2 is the shape parameter. Variable Y corresponds to the normal variable Z, where $Z = \ln(Y)$ and denoted by $Z \sim N\left(\mu_Z, \sigma_Z^2\right)$. The probability density function (*pdf*) of the Log-Normal distribution is as follows [12]:

$$f_Y\left(y\right) = \frac{1}{\sqrt{2\pi}\sigma_Z y} \exp\left(-\frac{\left(\ln\left(y\right) - \mu_Z\right)^2}{2\sigma_Z^2}\right) \quad (1)$$

where $y > 0, \; -\infty \le \mu_Z \le \infty, \; \sigma_Z > 0$.

The expected value and variance of Y is according to [13]:

$$\mu_Y = E\left(Y\right) = \exp\left(\mu_Z + \frac{\sigma_Z^2}{2}\right) \quad (2)$$

$$\sigma_Y^2 = Var(Y) = \exp\left(2\mu_Z + \sigma_Z^2\right)\left[\exp\left(\sigma_Z^2\right) - 1\right] \quad (3)$$

The LNR model can be written as follows [14]:

$$\mu_{Y_i} = E(Y_i) = \boldsymbol{\beta}_i^T \;;\quad i = 1, 2, \ldots, n \quad (4)$$

where $\mathbf{x}_i = \begin{bmatrix} 1 & x_{i1} & \cdots & x_{ip} \end{bmatrix}^T$ is a vector of predictor variable with dimensions $(p+1)\times 1$, $\boldsymbol{\beta} = \begin{bmatrix} \beta_0 & \beta_1 & \cdots & \beta_p \end{bmatrix}^T$ is a vector of regression coefficients with dimensions $(p+1)\times 1$.

2.2. Bivariate Log-Normal Distribution

The bivariate Log-Normal distribution is a combination of two continuous random variables with Log-Normal distribution respectively and correlate with each other [12]. Let Y_1, Y_2 is a random variable with bivariate Log-Normal distribution which is denoted by $Y_1, Y_2 \sim \mathrm{BLN}\left(\mu_{Z_1}, \mu_{Z_2}, \sigma_{Z_1}^2, \sigma_{Z_2}^2, \rho\right)$, then the joint pdf of Y_1, Y_2 is [13]:

$$f(y_1, y_2) = \frac{1}{2\pi y_1 y_2 \sigma_{Z_1}\sigma_{Z_2}\sqrt{1-\rho^2}} \exp\left(-\frac{q}{2}\right) \quad (5)$$

where $y_1, y_2 > 0, -1 < \rho < 1$, and

$$q = \frac{1}{1-\rho^2}\left[\left(\frac{\ln y_1 - \mu_{Z_1}}{\sigma_{Z_1}}\right)^2 - 2\rho\left(\frac{\ln y_1 - \mu_{Z_1}}{\sigma_{Z_1}}\right)\left(\frac{\ln y_2 - \mu_{Z_2}}{\sigma_{Z_2}}\right) + \left(\frac{\ln y_2 - \mu_{Z_2}}{\sigma_{Z_2}}\right)^2\right] \quad (6)$$

The expected value, variance, and correlation from bivariate Log-Normal distribution $Y_j,\ j = 1, 2$ is according to [16]:

$$\mu_{Y_j} = \exp\left(\mu_{Z_j} + \sigma_{Z_j}^2/2\right) \quad (7)$$

$$\sigma_{Y_j}^2 = \exp(2\mu_{Z_j} + \sigma_{Z_j}^2)\left[\exp(\sigma_{Z_j}^2) - 1\right]$$

$$\rho_{Y_1Y_2} = \frac{E\left[\left(Y_1 - \mu_{Y_1}\right)\left(Y_2 - \mu_{Y_2}\right)\right]}{\sigma_{Y_1}\sigma_{Y_2}} = \frac{\sigma_{Y_1Y_2}}{\sigma_{Y_1}\sigma_{Y_2}} = \frac{\exp\left(\sigma_{Z_1Z_2}\right) - 1}{\sqrt{\left[\exp\left(\sigma_{Z_1}^2\right) - 1\right]\left[\exp\left(\sigma_{Z_2}^2\right) - 1\right]}}$$

We highlight that in case of bivariate Log-Normal distribution, we involve more parameters than equation (1), that is $\mu_{Z_1}, \mu_{Z_2}, \sigma_{Z_1}^2, \sigma_{Z_2}^2,$ and ρ.

3. RESULTS AND DISCUSSION

BLNR is a regression of two response variables with a Log-Normal distribution with several predictor variables (X), the model is:

$$\mu_{Z_1 i} = \ln\left(\boldsymbol{\beta}_i^T \;{}_1\right) - \sigma_{Z_1}^2/2 \quad (8)$$

$$\mu_{Z_2 i} = \ln\left(\boldsymbol{\beta}_i^T \;{}_2\right) - \sigma_{Z_2}^2/2; \quad i = 1, 2, \ldots, n \quad (9)$$

So that the pdf of the BLNR model is obtained as follows:

$$f\left(y_{1i}, y_{2i}\right) = \begin{cases} \dfrac{1}{2\pi y_{1i} y_{2i} \sigma_{Z_1} \sigma_{Z_2} \sqrt{1-\rho^2}} \exp\left(-\dfrac{q_i}{2}\right) & , y_{1i}, y_{2i} > 0 \\ 0 & , y_{1i}, y_{2i} \le 0 \end{cases} \quad (10)$$

where $i = 1, 2, \ldots, n; \; -1 < \rho < 1$; and

$$q_i = \frac{1}{1-\rho^2}\left[\left(\frac{\ln y_{1i} - \left(\ln\left(\boldsymbol{\beta}_i^T \;{}_1\right) - \sigma_{Z_1}^2/2\right)}{\sigma_{Z_1}}\right)^2 + \left(\frac{\ln y_{2i} - \left(\ln\left(\boldsymbol{\beta}_i^T \;{}_2\right) - \sigma_{Z_2}^2/2\right)}{\sigma_{Z_2}}\right)^2 - 2\rho\left(\frac{\ln y_{1i} - \left(\ln\left(\boldsymbol{\beta}_i^T \;{}_1\right) - \sigma_{Z_1}^2/2\right)}{\sigma_{Z_1}}\right)\left(\frac{\ln y_{2i} - \left(\ln\left(\boldsymbol{\beta}_i^T \;{}_2\right) - \sigma_{Z_2}^2/2\right)}{\sigma_{Z_2}}\right)\right] .(11)$$

3.1. Parameter Estimation of Bivariate Log-Normal Regression (BLNR)

The parameter estimation of BLNR model has obtained using MLE method by maximizing the ln likelihood function. Parameter will be estimated is $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \sigma_{Z_1}^2, \sigma_{Z_2}^2$, and ρ. Then the likelihood and ln likelihood function of BLNR model from equation (10) are:

$$L\left(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \sigma_{Z_1}^2, \sigma_{Z_2}^2, \rho\right) = \prod_{i=1}^{n} \frac{1}{2\pi y_{1i} y_{2i} \sigma_{Z_1} \sigma_{Z_2} \sqrt{1-\rho^2}} \exp\left(-\frac{q_i}{2}\right) \quad (12)$$

$$Q = \ln L\left(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \sigma_{Z_1}^2, \sigma_{Z_2}^2, \rho\right) = -n \ln(2\pi) - \frac{n}{2}\left(\ln \sigma_{Z_1}^2 + \ln \sigma_{Z_2}^2 + \ln\left(1-\rho^2\right)\right) - \sum_{i=1}^{n} \ln y_{1i} y_{2i} - \frac{1}{2}\sum_{i=1}^{n} q_i \quad (13)$$

Maximizing the ln likelihood function is finding by the partial derivative of the parameter. The first derivative from the ln likelihood function of BLNR model equate to zero and produce an implicit form, then it continued with Newton Raphson iteration using the following algorithm:

1. Determine the initial value for parameter $\hat{\boldsymbol{\theta}}_{(0)} = \left[\hat{\boldsymbol{\beta}}_{1(0)}^T \;\; \hat{\boldsymbol{\beta}}_{2(0)}^T \;\; \hat{\sigma}_{Z_1(0)}^2 \;\; \hat{\sigma}_{Z_2(0)}^2 \;\; \hat{\rho}_{(0)}\right]^T$ from Log-Normal regression model of each response variable, and $\hat{\rho}_{(0)}$ is from correlation between two response variables.

2. Determine the gradient vector, whose elements consist of the first derivative of the ln likelihood function.

$$\mathbf{g}\left(\hat{\boldsymbol{\theta}}_{(m)}\right) = \begin{bmatrix} \frac{\partial Q}{\partial \boldsymbol{\beta}_1} & \frac{\partial Q}{\partial \boldsymbol{\beta}_2} & \frac{\partial Q}{\partial \sigma^2_{Z_1}} & \frac{\partial Q}{\partial \sigma^2_{Z_2}} & \frac{\partial Q}{\partial \rho} \end{bmatrix}^T$$

1. Determine the Hessian matrix, whose elements consist of the second derivative of the ln likelihood function.

$$\mathbf{H}\left(\hat{\boldsymbol{\theta}}_{(m)}\right) = \begin{bmatrix} \frac{\partial^2 Q}{\partial \boldsymbol{\beta}_1 \partial \boldsymbol{\beta}_1^T} & \frac{\partial^2 Q}{\partial \boldsymbol{\beta}_1 \partial \boldsymbol{\beta}_2^T} & \frac{\partial^2 Q}{\partial \boldsymbol{\beta}_1 \partial \sigma^2_{Z_1}} & \frac{\partial^2 Q}{\partial \boldsymbol{\beta}_1 \partial \sigma^2_{Z_2}} & \frac{\partial^2 Q}{\partial \boldsymbol{\beta}_1 \partial \rho} \\ & \frac{\partial^2 Q}{\partial \boldsymbol{\beta}_2 \partial \boldsymbol{\beta}_2^T} & \frac{\partial^2 Q}{\partial \boldsymbol{\beta}_2 \partial \sigma^2_{Z_1}} & \frac{\partial^2 Q}{\partial \boldsymbol{\beta}_2 \partial \sigma^2_{Z_2}} & \frac{\partial^2 Q}{\partial \boldsymbol{\beta}_2 \partial \rho} \\ & & \frac{\partial^2 Q}{\partial \sigma^2_{Z_1} \partial \sigma^2_{Z_1}} & \frac{\partial^2 Q}{\partial \sigma^2_{Z_1} \partial \sigma^2_{Z_2}} & \frac{\partial^2 Q}{\partial \sigma^2_{Z_1} \partial \rho} \\ & & & \frac{\partial^2 Q}{\partial \sigma^2_{Z_2} \partial \sigma^2_{Z_2}} & \frac{\partial^2 Q}{\partial \sigma^2_{Z_2} \partial \rho} \\ \text{symmetry} & & & & \frac{\partial^2 Q}{\partial \rho^2} \end{bmatrix}$$

The detail of every component for gradient vector and Hessian matrix is postpone in appendix.

1. Newton Raphson iteration starts from $m = 0$ in equation $\hat{\boldsymbol{\theta}}_{(m+1)} = \hat{\boldsymbol{\theta}}_{(m)} - \mathbf{H}^{-1}\left(\hat{\boldsymbol{\theta}}_{(m)}\right)\mathbf{g}\left(\hat{\boldsymbol{\theta}}_{(m)}\right)$, where $\hat{\boldsymbol{\theta}}_{(m)}$ is the parameter vector in the m^{th} iteration.

2. Repeat step (2) and so on for $m = m + 1$. The iteration will stop when $\left\|\hat{\boldsymbol{\theta}}_{(m+1)} - \hat{\boldsymbol{\theta}}_{(m)}\right\| < c$, where c is a very small positive number.

3.2. Hypothesis Testing of Bivariate Log-Normal Regression (BLNR)

Hypothesis testing of the BLNR parameters consists of simultaneous and partial testing. The statistics for hypothesis testing simultaneously using the MLRT method, and partial testing using the central limit theorem.

1. **Simultaneous Parameter Testing**

   Simultaneous testing is used to determine simultaneously the significance of the effect of the predictor variables on the response variables in the BLNR model with the following hypothesis:

$$H_0 : \beta_{11} = \beta_{12} = \cdots = \beta_{1p} = \beta_{21} = \beta_{22} = \cdots = \beta_{2p} = 0$$

H_1: at least one of $\beta_{jk} \neq 0;\ k = 1, 2, \cdots, p;\ j = 1, 2$

The test statistic in the MLRT method requires the maximum value of the ln likelihood function under H_0 $(\ln L(\omega))$ and under the population $(\ln L(\Omega))$. The parameter set under the population is $\Omega=\{\boldsymbol{\beta}_1,\boldsymbol{\beta}_2,\sigma^2_{Z_1},\sigma^2_{Z_2},\rho\}$ and the parameter set under H_0 is $\omega=\{\beta_{\omega10},\beta_{\omega20},\sigma^2_{\omega Z_1},\sigma^2_{\omega Z_2},\rho_\omega\}$.

The likelihood function under the population has been obtained in equation (12) and to obtain $L(\hat{\Omega})=\max_{\Omega} L(\Omega)$ it is done by maximizing the ln likelihood function in equation (13). The likelihood function under H_0 $(L(\omega))$ is as follows:

$$L(\omega)=\prod_{i=1}^{n}\frac{1}{2\pi y_{1i}y_{2i}\sigma_{\omega Z_1}\sigma_{\omega Z_2}\sqrt{1-\rho_\omega^2}}\exp\left(-\frac{q_{\omega i}}{2}\right) \quad (14)$$

$$q_{\omega i}=\frac{1}{\left(1-\rho_\omega^2\right)}\left[\left(\frac{\ln y_{1i}-\left(\ln\left(\beta_{\omega10}\right)-\frac{\sigma^2_{\omega Z_1}}{2}\right)}{\sigma_{\omega Z_1}}\right)^2+\left(\frac{\ln y_{2i}-\left(\ln\left(\beta_{\omega20}\right)-\frac{\sigma^2_{\omega Z_2}}{2}\right)}{\sigma_{\omega Z_2}}\right)^2-2\rho\left(\frac{\ln y_{1i}-\left(\ln\left(\beta_{\omega10}\right)-\frac{\sigma^2_{\omega Z_1}}{2}\right)}{\sigma_{\omega Z_1}}\right)\left(\frac{\ln y_{2i}-\left(\ln\left(\beta_{\omega20}\right)-\frac{\sigma^2_{\omega Z_2}}{2}\right)}{\sigma_{\omega Z_2}}\right)\right]$$

The ln likelihood function of equation (14) is:

$$Q^*=\ln L\left(\beta_{\omega10},\beta_{\omega20},\sigma^2_{Z_1},\sigma^2_{Z_2},\rho\right)=-n\ln(2\pi)-\frac{n}{2}\left(\ln\sigma^2_{\omega Z_1}+\ln\sigma^2_{\omega Z_2}+\ln\left(1-\rho_\omega^2\right)\right)-\sum_{i=1}^{n}\ln y_{1i}y_{2i}-\frac{1}{2}\sum_{i=1}^{n}q_{\omega i} \quad (15)$$

The parameter estimation under H_0 is obtained by maximizing the ln likelihood in equation (15) by finding the first partial derivative of each parameter under H_0 then equate to zero. Parameter estimation $\hat{\boldsymbol{\lambda}}=\left(\hat{\beta}_{\omega10}\ \ \hat{\beta}_{\omega20}\ \ \hat{\sigma}^2_{\omega Z_1}\ \ \hat{\sigma}^2_{\omega Z_2}\ \ \hat{\rho}_\omega\right)^T$ solved by *Newton Raphson* iteration. The detail of every component for gradient vector and Hessian matrix is postpone in appendix. After that, determine the value of the test statistic, as follows:

$$G^2=2\left[\ln L(\hat{\Omega})-\ln L(\hat{\omega})\right]\underset{n\to\infty}{\sim}\chi^2$$

$$=-n\left(\ln\hat{\sigma}^2_{Z_1}+\ln\hat{\sigma}^2_{Z_2}+\ln\left(1-\hat{\rho}^2\right)\right)-\sum_{i=1}^{n}\hat{q}_i+n\left(\ln\hat{\sigma}^2_{\omega Z_1}+\ln\hat{\sigma}^2_{\omega Z_2}+\ln\left(1-\hat{\rho}^2_\omega\right)\right)+\sum_{i=1}^{n}\hat{q}_{\omega i} \quad (16)$$

Reject H_0 if $G^2>\chi^2_{\alpha,p}$, where α is significance level and $p=n(\hat{\Omega})-n(\hat{\omega})$.

2. **Partial Testing**

Partial testing was conducted to determine the significance of each parameter in the model. The hypothesis for partial testing is as follows:

$$H_0:\beta_{jk}=0$$

$$H_1 : \beta_{jk} \neq 0;\ k = 1, 2, ..., p;\ j = 1, 2$$

The test statistics for the partial test of BLNR model is:

$$Z = \frac{\hat{\beta}_k}{se\left(\hat{\beta}_k\right)} \underset{n \to \infty}{\sim} N(0,1) \quad (17)$$

where $se\left(\hat{\beta}_k\right) = \sqrt{\hat{\text{var}}\left(\hat{\beta}_k\right)}$, and $\hat{\text{var}}\left(\hat{\beta}_k\right)$ is the main diagonal element of $-\mathbf{I}^{-1}\left(\hat{\ }\right)$. Reject H_0 if $|Z| > Z_{\alpha/2}$.

4. CONCLUSION

The BLNR model estimates the parameters for two correlated response variables simultaneously include the correlation coefficient between two variables, we involve more parameters than LNR. The estimation procedure is conducted by the standard MLE from pdf of bivariate Log-Normal distribution. This function is more complex than LNR, so we must focus and be careful when derivating the parameters to form the gradient vector and Hessian matrix in the Newton-Raphson iteration. For parameter testing, simultaneous test statistics on the BLNR model using MLRT, and the partial testing is derived from a central limit theorem which results in a Z-test statistic. The difference from parameter testing on univariate LNR previously is in the parameter being tested so that the hypothesis and calculation of the test statistics are different.

For further study, BLNR can be applied to a real case for example poverty gap index and poverty severity index. Spatial version of BLNR can be considered as an extension. This can be seen as poverty gap index and poverty severity index has spatial effect, so Geographically Weighted Regression based model such as Geographically Weighted Bivariate Log-Normal Regression (GWBLNR) can be derived. A varians of this model have been developed such as Bivariate Gamma [14], Three-Parameters Bivariate Gamma [15], Bivariate Weibull [16], Bivariate Generalized Poisson [17], Bivariate Zero Inflated Generalized Poisson [18], and Bivariate Zero Inflated Poisson Inverse Gaussian [19].

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