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Generalization of Goursat's Theorem for Subrings of Direct Products of n Rings

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Abstract.
Bauer et al. describe Goursat's theorem, representing the characteristics of subgroups of a direct product of two or more groups. In this paper, we expand into a ring structure that describes the characteristics of subrings of a direct product of rings. This research method is to analogize the evidence by Bauer et al. in the group for generalization. In our main results, every subring of the direct product of rings is determined by ring epimorphism between the ring and factor ring.

Keywords: Goursat's theorem, subrings, rings

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1. INTRODUCTION

Edouard Goursat [1], a French mathematician, discovered Goursat's Theorem, which describes the characteristics of subgroups of the direct product of two groups, i.e., an isomorphism between factor groups of subgroups of the given groups determines each subgroup of the direct product of two groups. Give an instance of applying the Goursat's Theorem to determine the subgroups of $G_1 \times G_2$, to be counted the number of subgroups of $S_3 \times S_3$, and to prove the Lemma Zassenhaus [2]. They also describe how can state Goursat's Theorem in the context of rings, ideals, subrings and modules. Goursat's Theorem to give explicit formulas for a total number of subgroups of $Z_m \times Z_n$ and a total number of subrings of $Z_m \times Z_n$ [3, 4]. This theorem is further applied to provide an exact formula for the total number of subgroups of a finite abelian *p*-group $Z_{p^m} \times Z_{p^n} \times Z_{p^l}$ [5]. Then extends Goursat's Theorem to *R*-module [6], whereas expand to *R*-algebraic [7].

Other researchers have extensively used Goursat's Theorem to advance algebra [3–5, 8–15]. Therefore, this theorem is important to be studied in more depth; for example, Bauer et al. [12] generalize to a direct product of groups by designing an asymmetric version of Goursat's Theorem for the two groups and then applying recursively. Then extends to a direct product of modules [16]. However, to the best of our knowledge, there has been no research on expanding the Goursat Theorem to the direct product of n rings. This theorem is critical for following researchers to examine and use, particularly in ring structures. Therefore, this study aimed to investigate the Goursat Theorem's extension to the direct product of n rings.

2. RESEARCH METHOD

This research analogizes the findings obtained in the group to generalize to rings by devising an asymmetric version of Goursat's Theorem for two rings that then apply recursively [12].

3. RESULT AND DISCUSSION

Let A_1 and A_2 be rings with zero elements 0_{A_i} respectively, I_1 is an ideal of A_1 , and I_2 is an ideal of A_2. The graph of ring homomorphism $f : A_1 \to A_2$, denoted by G_f, is the set $\{(a_1, a_2) \in A_1 \times A_2 f(a_1) = a_2\}$. It is clear immediate that G_f is a subring of $A_1 \times A_2$. Associated to $A_1 \times A_2$ there are some natural homomorphisms: $\pi_1 : A_1 \times A_2 \to A_1$, $\pi_2 : A_1 \times A_2 \to A_2$, $\iota_1 : A_1 \to A_1 \times A_2$, $\iota_2 : A_2 \to A_1 \times A_2$, $\rho : A_1 \times A_2 \to A_1 \times \frac{A_2}{I_2}$, and $\rho_0 : A_1 \times A_2$
$\to \frac{A_1}{I_1} \times \frac{A_2}{I_2}$. Here, we write $A_1 \cong^f A_2$ to denote that A_1 and A_2 are isomorphic via a ring isomorphism f.

Theorem 1. *Let A_1 and A_2 be rings, R is a subring of $A_1 \times A_2$. Then $\frac{\pi_1(R)}{\iota_1^{-1}(R)} \cong^f \frac{\pi_2(R)}{\iota_2^{-1}(R)}$ and $\rho_0{}^{-1}(G_f) = R$.*

Proof. Define a map $f : \frac{\pi_1(R)}{\iota_1^{-1}(R)} \to \frac{\pi_2(R)}{\iota_2^{-1}(R)}$ by $f(a_1 + \iota_1{}^{-1}(R)) := a_2 + \iota_2{}^{-1}(R)$ when $(a_1, a_2) \in R$. Suppose that $\frac{a_1+\iota_1{}^{-1}(R), b_1+\iota_1{}^{-1}(R) \in \pi_1(R)}{\iota_1{}^{-1}(R)}$ with $a_1 + \iota_1{}^{-1}(R) = b_1 + \iota_1{}^{-1}(R)$. Then $a_1 - b_1 \in \iota_1{}^{-1}(R)$. Since $a_1 \in \pi_1(R)$ and $b_1 \in \pi_1(R)$, $(a_1, x), (b_1, y) \in R$ for some $x, y \in A_2$, whence $f(a_1 + \iota_1{}^{-1}(R)) = x + \iota_2{}^{-1}(R)$ and $f(b_1 + \iota_1{}^{-1}(R)) = y + \iota_2{}^{-1}(R)$. Thus $(0_{A_1}, x - y) = (a_1, x) - (b_1, y) - (a_1 - b_1, 0_{A_2}) \in R$ since R is a subring of $A_1 \times A_2$. Therefore, $x - y \in \iota_2{}^{-1}(R)$ and hence $x + \iota_2{}^{-1}(R) = y + \iota_2{}^{-1}(R)$. Verify that f is a ring epimorphism. Finally, let $p_1 + \iota_1{}^{-1}(R), q_1 + \iota_1{}^{-1}(R) \in \frac{\pi_1(R)}{\iota_1{}^{-1}(R)}$ with $f(p_1 + \iota_1{}^{-1}(R)) = f(q_1 + \iota_1{}^{-1}(R))$. Since $p_1 \in \pi_1(R)$ and $q_1 \in \pi_1(R)$ $(p_1, p_2), (q_1, q_2) \in R$ for some $p_2, q_2 \in A_2$, so that $p_2 + \iota_2{}^{-1}(R) = q_2 + \iota_2{}^{-1}(R)$. Consequently, $(p_1 - q_1, 0_{A_2}) = (p_1, p_2) - (q_1, q_2) - (0_{A_1}, p_2 - q_2) \in R$ since R is a subring of $A_1 \times A_2$.

Suppose that $(a_1, a_2) \in \rho_0^{-1}(G_f)$. Then $(a_1 + \iota_1^{-1}(R), a_2 + \iota_2^{-1}(R)) = \rho_0(a_1, a_2) \in G_f$, so that $f(a_1 + \iota_1^{-1}(R)) = a_2 + \iota_2^{-1}(R)$. Thus $(a_1, a_2) \in R$. Let $(b_1, b_2) \in R$. Then $b_1 \in \pi_1(R)$ and $b_2 \in \pi_2(R)$. Therefore, $\rho_0(b_1, b_2) = (b_1 + \iota_1^{-1}(R), b_2 + \iota_2^{-1}(R)) \in G_f$. Thus $(b_1, b_2) \in \rho_0^{-1}(G_f)$.

Theorem 2. *Let A_1 and A_2 be rings, I_i is an ideal of A_i. If $\frac{A_1}{I_1} \cong^f \frac{A_2}{I_2}$ then (i)$\pi_1(\rho_0^{-1}(G_f)) = A_1$, (ii) $\pi_2(\rho_0^{-1}(G_f)) = A_2$, (iii) $\iota_1^{-1}(\rho_0^{-1}(G_f)) = I_1$, and (iv) $\iota_2^{-1}(\rho_0^{-1}(G_f)) = I_2$.*

Proof. (i) Clearly $\pi_1(\rho_0^{-1}(G_f)) \subseteq A_1$. Suppose that $a_1 \in A_1$. There exists an element $a_2 \in A_2$ such that $(a_1 + I_1) = a_2 + I_2$. Hence $\rho_0(a_1, a_2) = (a_1 + I_1, a_2 + I_2) \in G_f$, whence $(a_1, a_2) \in \rho_0^{-1}(G_f)$ and we get $a_1 \in \pi_1(\rho_0^{-1}(G_f))$. (ii) Suppose that $a_2 \in A_2$. There exists an element $a_1 \in A_1$ such that $f(a_1 + I_1) = a_2 + I_2$ since f is surjective. Thus $a_2 \in \pi_2(\rho_0^{-1}(G_f))$. (iii) Suppose $a_1 \in \iota_1^{-1}(\rho_0^{-1}(G_f))$. Then $\rho_0(a_1, 0_{A_2}) \in G_f$ and hence $f(a_1 + I_1) = I_2 = f(I_1)$. Since f is injective, $a_1 + I_1 = I_1$. Therefore, $a_1 \in I_1$. Conversely, if $b_1 \in I_1$, then $b_1 + I_1 = I_1$. Thus $(b_1 + I_1) = f(I_1) = I_2$. Consequently, $\rho_0(b_1, 0_{A_2}) = (b_1 + I_1, I_2) \in G_f$. Therefore, $b_1 \in \iota_1^{-1}(\rho_0^{-1}(G_f))$. (iv) is proved similarly. ⊠

Definition 3. Let $A_1, A_2, \dots, A_n$ be a finite collection of rings and let R be a nonempty subset of $A_1 \times A_2 \times \cdots \times A_n$. If $S \subset \hat{n} := \{1, 2, \dots, n\}$ and $j \in \hat{n} - S$, let

$$R(jS) := \{a_j \in A_j (a_1, a_2, \dots, a_n) \in R \; for\ some\ a_i \in A_i,\ i \in \hat{n} - \{j\} \; with\ a_i = 0_{A_i}\ if\ i \in S\}$$

For example, $R(2\hat{n} - \{2\}) := \{a_2 \in A_2 (0_{A_1}, a_2, 0_{A_3}, 0_{A_4}, \dots, 0_{A_n}) \in R\}$ and $R(1\{4, 5, \dots, n\}) := \{a_1 \in A_1 (a_1, a_2, a_3,$

$0_{A_4}, 0_{A_5}, \dots, 0_{A_n}) \in R \; for\ some\ a_2 \in A_2,\ a_3 \in A_3\}$. Let R be a subring of $A_1 \times A_2$. By Theorem 1, $\frac{R(1\emptyset)}{R(1\{2\})} \cong^f \frac{R(2\emptyset)}{R(2\{1\})}$ since $\pi_1(R) = R(1\emptyset)$, $\pi_2(R) = R(2\emptyset)$, $\iota_1^{-1}(R) = R(1\{2\})$, and $\iota_2^{-1}(R) = R(2\{1\})$. Let A_1, $A_2, \dots, A_n$ be a finite collection of rings, and let π_i : $A_1 \times A_2 \times \cdots \times A_n \to A_i$ be given by $\pi_i(a_1, a_2, \dots, a_n) = a_i$, $i \in \hat{n}$, π_i is called the standard projection onto the ith factor. Of course $\pi_i(R) = R(i\emptyset)$ for all subring $\subseteq A_1 \times A_2 \times \cdots \times A_n$. Now we define $\prod_i : A_1 \times A_2 \times \cdots \times A_n \to A_1 \times A_2 \times \cdots \times A_i$ by $\prod_i(a_1, a_2, \dots, a_n) = (a_1, a_2, \dots, a_i)$, $i \in \hat{n}$, $\prod_i$ is called the standard projection onto the first i factors (e.g., $\prod_1 = \pi_1$ and $\prod_n = id_{A_1 \times A_2 \times \cdots \times A_n}$).

Theorem 4. *Let $A_1, A_2, \dots, A_n$ be a finite collection of rings and $j \in \hat{n} - S$. If R is a subring of $A_1 \times A_2 \times \cdots \times A_n$ then $R(jS)$ is a subring of A_j and $R(jS)$ is an ideal of $R(jT)$ with $T \subseteq S$.*

Proof. Let $a, b \in R(jS)$. Then $a = a_j$ and $b = b_j$ such that $(a_1, a_2, \dots, a_j, \dots, a_n)$, $(b_1, b_2, \dots, b_j, \dots, b_n) \in R$ where $a_i = b_i = 0_{A_i}$ $(i \in S)$ and for some $a_i, b_i \in A_i$ $(i \in \hat{n} - S \cup \{j\})$. Since R is a subring of $A_1 \times A_2 \times \cdots \times A_n$, $(a_1 - b_1,\ a_2 - b_2, \dots, a_n - b_n), (a_1 b_1, a_2 b_2, \dots, a_n b_n) \in R$. We have $a - b = a_j - b_j \in R(jS)$ and $ab = a_j b_j \in R(jS)$, because $a_i - b_i = a_ib_i = 0_{A_i}$ $(i \in S)$ and there exists $a_i - b_i, a_ib_i \in A_i$ $(i \in \hat{n} - S \cup \{j\})$. Therefore, $R(jS)$ is a subring of A_j. Note that $R(jS)$ is a subring of $R(jT)$ since both $R(jS)$ and $R(jT)$ are subring of A_j and $(jS) \subseteq R(jT)$. Suppose $u \in R(jS)$ and $r \in R(jT)$. Then $u = u_j$ and $r = r_j$ such that $(u_1, u_2, \dots, u_j, \dots, u_n), (r_1, r_2, \dots, r_j, , \dots, r_n) \in R$ where component $u_i = 0_{A_i}$ $(i \in S)$ and $r_i = 0_{A_i}$ $(i \in T)$; for some $u_i \in A_i$ $(i \in \hat{n} - S \cup \{j\})$ and $r_i \in A_i$ $(i \in \hat{n} - T \cup \{j\})$. Since R is a subring of $A_1 \times A_2 \times \cdots \times A_n$, $(u_1r_1, u_2r_2, \dots, u_jr_j, \dots, u_nr_n) \in R$. Furthermore, $ur = u_jr_j \in R(jS)$ since $u_ir_i = 0_{A_i}r_i = 0_{A_i}$ $(i \in S)$ and there exists $u_ir_i \in A_i$ $(i \in \hat{n} - S \cup \{j\})$. Thus $R(jS)$ is a right ideal of $R(jT)$. The proof for $R(jS)$ is a left ideal of $R(jT)$ is similar. ☐

Theorem 5. *Let $A_1, A_2, \dots, A_n$ be a finite collection of rings. If R is a subring of $A_1 \times A_2 \times \cdots \times A_n$ then $f_k : \prod_k(R) \to \frac{R(k+1\emptyset)}{R(k+1\hat{k})}$ $(1 \le k < n)$ is a ring epimorphism.*

Proof. Define a map $f_k : \prod_k(R) \to \frac{R(k+1\emptyset)}{R(k+1\hat{k})}$ by $f_k(a_1, a_2, \dots, a_k) := a_{k+1} + R(k + 1\hat{k})$ when $(a_1, a_2, a_k, a_{k+1}, \dots, a_n) \in R$ for some $a_j \in A_j$ $(k + 1 < j \le n)$. Let $p = \prod_k(p_1, p_2, \dots, p_k, \dots, p_n) \in \prod_k(R)$ and $q = \prod_k(q_1, q_2, \dots, q_k, \dots, q_n) \in \prod_k(R)$ with $p = q$. Then $p_j = q_j$ for all $j \in \hat{k}$. Since $(p_1, p_2, \dots, p_n), (q_1, q_2, \dots, q_n) \in R$, $f_k(p_1, p_2, \dots, p_k) = p_{k+1} + R(k+1\hat{k})$ and $f_k(q_1, q_2, \dots, q_k) = q_{k+1} + R(k+1\hat{k})$. But $(0_{A_1}, 0_{A_2}, \dots, 0_{A_k}, p_{k+1} - q_{k+1}, \dots, p_n - q_n) = (p_1, p_2, \dots, p_k, p_{k+1}, \dots, p_n) - (q_1, q_2, \dots, q_k, q_{k+1}, \dots, q_n) \in R$. Therefore, $p_{k+1} - q_{k+1} \in R(k + 1\hat{k})$. Hence, $p_{k+1} + R(k + 1\hat{k}) = q_{k+1} + R(k + 1\hat{k})$, so f_k is well-defined. Verify that f_k is a surjective ring homomorphism. ☐

Theorem 6. *Let $A_1, A_2, \dots, A_n$ be a finite collection of rings. If R is a subring of $A_1 \times A_2 \times \cdots \times A_n$ and sequence $\{\Lambda_i\}_{i=1}^{i=n}$ that satisfies the recurrence relation $\Lambda_i = \rho^{-1}(G_{f_{i-1}})$ where $f_{i-1} : \Lambda_{i-1} \to \frac{R(i\emptyset)}{R(i\underline{i-1})}$ with initial conditions $\Lambda_1 := R(1\emptyset)$, then $\Lambda_j = \Pi_j(R)$ for all $j \ge 2$* and *f_j is a ring epimorphism for all $j < n$.*

Proof. We will use induction on j. For $j = 2$. The map $f_1 : \Lambda_1 := R(1\emptyset) = \pi_1(R) = \Pi_1(R) \to \frac{R(2\emptyset)}{R(2\hat{1})}$ is a ring epimorphism by Theorem 5. Now we claim that $\rho^{-1}(G_{f_1}) = \Pi_2(R)$. If $(a_1, a_2) \in \rho^{-1}(G_{f_1})$, then $(a_1, a_2 + +R(2\hat{1})) = \rho(a_1, a_2) \in G_{f_1}$ and hence $f_1(a_1) = a_2 + R(2\hat{1})$. Consequently, $(a_1, a_2, \dots, a_n) \in R$ for some $a_j \in A_j$ $(2 < j \le n)$. But $(a_1, a_2) = \Pi_2(a_1, a_2, \dots, a_n) \in \Pi_2(R)$ and $(a_1, a_2) \in \rho^{-1}(G_{f_1})$ imply $\rho^{-1}(G_{f_1}) \subseteq \Pi_2(R)$. Con-versely, let $\Pi_2(b_1, b_2, \dots, b_n) \in \Pi_2(R)$. Since $(b_1, b_2, \dots, b_n) \in R$, we have $f_1(b_1) = b_2 + R(2\hat{1})$. Hence $(b_1, b_2) = (b_1, b_2 + R(2\hat{1})) \in G_{f_1}$. Thus $\Pi_2(b_1, b_2, \dots, b_n) = (b_1, b_2) \in \rho^{-1}(G_{f_1})$ and we conclude that $\Pi_2(R) \subseteq \rho^{-1}(G_{f_1})$. Therefore, $\Lambda_2 = \rho^{-1}(G_{f_1}) = \Pi_2(R)$.

Now suppose that $\Lambda_k = \Pi_k(R)$ and assume the result is true for all $k \ge 2$. Claim that $\rho^{-1}(G_{f_k}) = \Pi_{k+1}(R)$, where $f_k : \Lambda_k = \prod_k(R) \to \frac{R(k+1\emptyset)}{R(k+1\hat{k})}$ is a ring epimorphism by Theorem 5. Let $u \in \rho^{-1}(G_{f_k})$. Then $u = (a_1, a_2, \dots, a_k, a_{k+1}) \in \Lambda_k \times R(k + 1\emptyset)$ such that $(a_1, a_2, \dots, a_k, a_{k+1} + R(k + 1\hat{k})) = \rho(a_1, a_2, \dots, a_k, a_{k+1}) \in G_{f_k}$ for some elements $(a_1, a_2, \dots, a_k) \in \Lambda_k$ and $a_{k+1} \in R(k + 1\emptyset)$. Hence, $f_k(a_1, a_2, \dots, a_k) = a_{k+1} + R(k +$

$1\hat{k}$). So, $(a_1, a_2, \dots, a_k, a_{k+1}, a_{k+2}, \dots, a_n) \in R$ for some elements $a_j \in A_j$ $(k+1 < j \leq n)$. Thus $u = (a_1, a_2, \dots, a_k, a_{k+1}) = \Pi_{k+1}(a_1, a_2, \dots, a_k, a_{k+1}, a_{k+2}, \dots, a_n) \in \Pi_{k+1}(R)$and we conclude that $\rho^{-1}(G_{f_k}) \subseteq \Pi_{k+1}(R)$. Next, let $v \in \Pi_{k+1}(R)$. Then $v = \Pi_{k+1}(b_1, b_2, \dots, b_n)$ for some element $(b_1, b_2, \dots, b_n) \in R$ so that $b_{k+1} \in R(k+1\emptyset)$. By formula f_k, we have $f_k(b_1, b_2, \dots, b_k) = b_{k+1} + R(k+1\hat{k})$. Therefore $\rho(b_1, b_2, \dots, b_k, b_{k+1}) = (b_1, b_2, \dots, b_k, b_{k+1} + R(k+1\hat{k})) \in G_{f_k}$. Consequently $v = (b_1, b_2, \dots, b_k, b_{k+1}) \in \rho^{-1}(G_{f_k})$. So, $\Pi_{k+1}(R) \subseteq \rho^{-1}(G_{f_k})$ and the claim is verified. We conclude deduce that $\Lambda_{k+1} = \rho^{-1}(G_{f_k}) = \Pi_{k+1}(R)$. ☒

Theorem 7. *Let $A_1, A_2, \dots, A_n$ be a finite collection of rings, I_i is an ideal of A_i $(i \neq 1)$ and sequence $\{\Lambda_i\}_{i=1}^{i=n}$ that satisfies the recurrence relation $\Lambda_i = \rho^{-1}(G_{f_{i-1}}) \subseteq A_1 \times A_2 \times \cdots \times A_i$ where $f_{i-1} : \Lambda_{i-1} \to \frac{A_i}{I_i}$ is a ring epimorphism with initial conditions $\Lambda_1 := A_1$. Then $\Lambda_n(i\emptyset) = A_i$ $(1 \leq i \leq n)$ and $\Lambda_n(i\underline{i-1}) = I_i$ $(1 < i \leq n)$.*

Proof. Clearly $\Lambda_n(i\emptyset) \subseteq A_i$. Let $a_i \in A_i$. Since $f_{i-1} : \Lambda_{i-1} \to \frac{A_i}{I_i}$ is surjective, there exists $(a_1, a_2, \dots, a_{i-1}) \in \Lambda_{i-1}$ such that $f_{i-1}(a_1, a_2, \dots, a_{i-1}) = a_i + I_i$. Hence, $\rho(a_1, a_2, \dots, a_{i-1}, a_i) \in G_{f_{i-1}}$ so that $(a_1, a_2, \dots, a_i) \in \rho^{-1}(G_{f_{i-1}}) = \Lambda_i$. Since $f_i : \Lambda_i \to \frac{A_{i+1}}{I_{i+1}}$ is a function, there is $a_{i+1} \in A_{i+1}$ such that $f_i(a_1, a_2, \dots, a_i) = a_{i+1} + I_{i+1}$. Which implies that $(a_1, a_2, \dots, a_{i+1}) \in \rho^{-1}(G_{f_i}) = \Lambda_{i+1}$. Likewise, $(a_1, a_2, \dots, a_{i+2}) \in \Lambda_{i+2}, (a_1, a_2, \dots, a_{i+3}) \in \Lambda_{i+3}, \dots, (a_1, a_2, \dots, a_n) \in \Lambda_n$ for some $a_{i+2} \in A_{i+2}, a_{i+3} \in A_{i+3}, \dots, a_n \in A_n$. Thus $a_i \in \Lambda_n(i\emptyset)$ and hence $A_i \subseteq \Lambda_n(i\emptyset)$.

Observe that $\Lambda_n(i\underline{i-1}) \subseteq I_i$. Suppose $a_i \in \Lambda_n(i\underline{i-1}) = \rho^{-1}(G_{f_{n-1}})(i\underline{i-1})$. By Definition 3, $(0_{A_1}, 0_{A_2}, \dots, 0_{A_{i-1}}, a_i, \dots, a_n) \in \rho^{-1}(G_{f_{n-1}})$ for some $a_j \in A_j$ $(i < j \leq n)$. Therefore, we have $f_{n-1}(0_{A_1}, 0_{A_2}, \dots, 0_{A_{i-1}}, a_i, \dots, a_{n-1}) = a_n + I_n$. Since Λ_{n-1} is domain of f_{n-1}, $(0_{A_1}, 0_{A_2}, \dots, 0_{A_{i-1}}, a_i, \dots, a_{n-1}) \in \Lambda_{n-1} = \rho^{-1}(G_{f_{n-2}})$. Hence, we have $f_{n-2}(0_{A_1}, 0_{A_2}, \dots, 0_{A_{i-1}}, a_i, \dots, a_{n-2}) = a_{n-1} + I_{n-1}$ so that $(0_{A_1}, 0_{A_2}, \dots, 0_{A_{i-1}}, a_i, \dots, a_{n-2}) \in \Lambda_{n-2} = \rho^{-1}(G_{f_{n-3}})$ since Λ_{n-2} is domain of f_{n-2}. Similarly, $(0_{A_1}, 0_{A_2}, \dots, 0_{A_{i-1}}, a_i) \in \Lambda_i = \rho^{-1}(G_{f_{i-1}})$. Consequently, $f_{i-1}(0_{A_1}, 0_{A_2}, \dots, 0_{A_{i-1}}) = a_i + I_i$. So, $I_i = a_i + I_i$ since f_{i-1} is a ring homomorphism. Thus we have $a_i \in I_i$. Next, let $b_i \in I_i$. Then $I_i = b_i + I_i$. Since f_{i-1} is a ring homomorphism, $f_{i-1}(0_{A_1}, 0_{A_2}, \dots, 0_{A_{i-1}}) = I_i = b_i + I_i$, whence $\rho(0_{A_1}, 0_{A_2}, \dots, 0_{A_{i-1}}, b_i) \in G_{f_{i-1}}$. Hence $(0_{A_1}, 0_{A_2}, \dots, 0_{A_{i-1}}, b_i) \in \rho^{-1}(G_{f_{i-1}}) = \Lambda_i$. Since $f_i : \Lambda_i \to \frac{A_{i+1}}{I_{i+1}}$ is a function, there exists $b_{i+1} \in A_{i+1}$ such that $f_i(0_{A_1}, 0_{A_2}, \dots, 0_{A_{i-1}}, b_i) = b_{i+1} + I_i$. Thus $(0_{A_1}, 0_{A_2}, \dots, 0_{A_{i-1}}, b_i, b_{i+1}) \in \rho^{-1}(G_{f_i}) = \Lambda_{i+1}$. Like-wise, $(0_{A_1}, 0_{A_2}, \dots, 0_{A_{i-1}}, b_i, b_{i+1}, b_{i+2}) \in \Lambda_{i+2}$, $(0_{A_1}, 0_{A_2}, \dots, 0_{A_{i-1}}, b_i, b_{i+1}, b_{i+2}, b_{i+3}) \in \Lambda_{i+3}, \dots, (0_{A_1}, 0_{A_2}, \dots, 0_{A_{i-1}}, b_i, b_{i+1}, \dots, b_n) \in \Lambda_n$ for some $b_{i+2} \in A_{i+2}, b_{i+3} \in A_{i+3}, \dots, b_n \in A_n$. Consequently $b_i \in \Lambda_n(i\underline{i-1})$.

Here is Goursat's Theorem starting from $n = 2$ (Theorem 8), $n = 3$ (Theorem 10), and $n = 4$ (Theorem 11). We will state the generalization of Goursat's Theorem in Theorem 12 for $n \geq 2$.

Theorem 8. [2] *(Goursat's Theorem for subrings of a direct product of 2 rings) Let A_1 and A_2 be rings, S is the set of all subrings of $A_1 \times A_2$, and T is the set of all 5-tuples (W_1, S_1, W_2, S_2, f) where S_i is an ideal of W_i, W_i is a subring of A_i $(i = 1, 2)$ and $\frac{W_1}{S_1} \cong^f \frac{W_2}{S_2}$. Then there is a one-to-one correspondence between S and T.*

Proof. Let $R \in S$. Then by Theorem 1 and 4, $V = (R(1\varnothing), R(1\{2\}), R(2\varnothing), R(2\{1\}), f) \in T$. Now we define $\hat{\alpha} : S \to T$ by $\hat{\alpha}(R) := V$. Conversely, for an arbitrary $Q_5 = (W_1, S_1, W_2, S_2, f) \in T$ the map $\hat{\beta} : T \to S$ defined by $\hat{\beta}(Q_5) := \rho_0^{-1}(G_f)$. We must show $\hat{\alpha}$ and $\hat{\beta}$ are mutually inverse. Let $R \in S$. Then by Theorem 1, $\hat{\beta}(\hat{\alpha}(R)) = \hat{\beta}(R(1\varnothing), R(1\{2\}), R(2\varnothing), R(2\{1\}), f) = \rho_0^{-1}(G_f) = R$. Next, let $Q = (W_1, S_1, W_2, S_2, f) \in T$. Then by Theo- rem 2, $\hat{\alpha}(\hat{\beta}(Q)) = \hat{\alpha}(\rho_0^{-1}(G_f)) = ((\rho_0^{-1}(G_f)(1\varnothing), \rho_0^{-1}(G_f)(1\{2\}), \rho_0^{-1}$ $(G_f)(2\varnothing), \rho_0^{-1}(G_f)(2\{1\}), g) = Q$.

Theorem 9. *(Asymmetric version of Goursat's Theorem for subrings of a direct product of 2 rings) Let A_1 and A_2 be rings, S is the set of all subrings of $A_1 \times A_2$, and T_4 is the set of all 4-tuples (W_1, W_2, S_2, f_1) where S_2 is an ideal of W_2, W_i is a subring of A_i $(i = 1, 2)$, and $f_1 : W_1 \to \frac{W_2}{S_2}$ is a ring epimorphism. Then there is a one-to-one correspondence between S and T_4.*

Proof. Suppose $R \in S$. Then 4-tuple $(R(1\varnothing), R(2\varnothing), R(2\{1\}), f_1) \in T_4$ since $R(2\{1\})$ is an ideal of $R(2\varnothing)$, $R(1\varnothing)$ is a subring of A_i (by Theorem 4), and $f_1 : R(1\varnothing) \to \frac{R(2\varnothing)}{R(2\{1\})}$ is a ring epimorphism (by Theorem 5). Define a map $\alpha_2 : S \to T_4$ by $\alpha_2(R) := (R(1\varnothing), R(2\varnothing), R(2\{1\}), f_1)$. Conversely, define a function $\beta_2 : T_4 \to S$ by $\beta_2(Q_4) := \rho^{-1}(G_{f_1})$ for all $Q_4 = (W_1, W_2, S_2, f) \in T_4$. Now, let $R \in S$. Then by Theorem 5, there exists a ring epimorphism $f_1 : R(1\varnothing) \to \frac{R(2\varnothing)}{R(2\{1\})}$. By Theorem 6, $\beta_2(\alpha(R)) = \beta_2(R(1\varnothing), R(2\varnothing), R(2\{1\}), f_1) = \rho^{-1}(G_{f_1}) = \Lambda_2 = \Pi_2(R) = R$. Finally, suppose that $Q = (W_1, W_2, S_2, f_1) \in T_4$. Then by Theorem 7, $\alpha_2(\beta_2(Q)) = \alpha_2(\rho^{-1}(\mathcal{G}_{f_1})) = (\rho^{-1}(\mathcal{G}_{f_1})(1|\varnothing), \rho^{-1}(\mathcal{G}_{f_1})(2|\varnothing), \rho^{-1}(\mathcal{G}_{f_1})(2|\{1\}), g_1) = (\;_2(|\varnothing) \;_2(|\varnothing)$

$\Lambda_2(2\{1\}), g_1) = Q$. The maps α_2 and β_2 are inverse to each other.

Theorem 10. *(Goursat's Theorem for subrings of a direct product of 3 rings) Let A_1, A_2, and A_2 be rings, S is the set of all subrings of $A_1 \times A_2 \times A_3$, and T_7 is the set of all 7-tuples $(W_1, W_2, S_2, f_1, W_3, S_3, f_2)$ where S_j is an ideal of W_j $(j \neq 1)$, W_i is a subring of A_i, both $f_1 : W_1 \to \frac{W_2}{S_2}$ and $f_2 : \beta_2(W_1, W_2, S_2, f_1) \to \frac{W_3}{S_3}$ are ring epimorphisms with β_2 as defined in Theorem 9. Then there is a one-to-one correspondence S and T_7.*

Proof. Let $R \in S$. Then according to Theorem 4 and 6, $(R(1\varnothing), R(2\varnothing), R(2\hat{1}), f_1, R(3\varnothing), R(3\hat{2}), f_2) \in T_7$. Thus define a function $\alpha_3 : S \to T_7$ by $\alpha_3(R) := (R(1\varnothing), R(2\varnothing), R(2\hat{1}), f_1, R(3\varnothing), R(3\hat{2}), f_2)$. Conversely, suppose that $Q_7 = (W_1, W_2, S_2, f_1, W_3, S_3, f_2) \in T_7$. Define a function $\beta_3 : T_7 \to S$ by $\beta_3(Q_7) := \rho^{-1}(G_{f_2})$. Next, we prove α_3

and β_3 are inverse to each other. Let $R \in S$. Then by Theorem 6, $\beta_3(\alpha_3(R)) = \beta_3(R(1\emptyset), R(2\emptyset), R(2\hat{1}), f_1, R(3\emptyset), R(3\hat{2}), f_2) = \rho^{-1}(G_{f_2}) = \Lambda_3 = \Pi_3(R) = R$. Finally, let $Q_7 = (W_1, W_2, S_2, f_1, W_3, S_3, f_2) \in T_4$. Then by Theorem 7, we have $\alpha_3(\beta_3(Q_7)) = \alpha_3(\rho^{-1}(G_{f_2})) = (\rho^{-1}(G_{f_2})(1\emptyset), \rho^{-1}(G_{f_2})(2\emptyset), \rho^{-1}(G_{f_2})(2\hat{1}), g_1, \rho^{-1}(G_{f_2})(3\emptyset), \rho^{-1}(G_{f_2})(3\hat{2}), g_2) = (\Lambda_3(1\emptyset), \Lambda_3(2\emptyset), \Lambda_3(2\hat{1}), g_1, \Lambda_3(3\emptyset), \Lambda_3(3\hat{2}), g_2) = Q_7$.

Theorem 11. *(Goursat's Theorem for subrings of a direct product of 4 rings) Let A_1, A_2, A_3, and A_4 be rings, S is the set of all subrings of $A_1 \times A_2 \times A_3 \times A_4$, and T_{10} is the set of all 10-tuples $(W_1, W_2, S_2, f_1, W_3, S_3, f_2, W_4, S_4, f_3)$ where S_j is an ideal of W_j $(j \neq 1)$, W_i is a subring of A_i, $f_1 : W_1 \to \frac{W_2}{S_2}$, $f_2 : \beta_2(W_1, W_2, S_2, f_1) \to \frac{W_3}{S_3}$, and $f_3 : \beta_2(\beta_2(W_1, W_2, S_2, f_1), W_3, S_3, f_2) \to \frac{W_4}{S_4}$ with f_i is a ring epimorphism and β_2 as defined in Theorem 9. Then there is a one-to-one correspondence S and T_{10}.*

Proof. Define the mapping $\alpha_4 : S \to T_{10}$ by $\alpha_4(R) := (R(1\emptyset), R(2\emptyset), R(2\hat{1}), f_1, R(3\emptyset), R(3\hat{2}), f_2, R(4\emptyset), R(4\hat{3}), f_3) \in T_{10}$ for all $R \in S$; and $\beta_4 : T_{10} \to S$ by $\beta_4(Q_{10}) := \rho^{-1}(G_{f_3}) \in S$ for all $Q_{10} = (W_1, W_2, S_2, f_1, W_3, S_3, f_2, W_4, S_4, f_3) \in T_{10}$. Applying Theorem 6 and 7, we conclude that α_4 and β_4 are inverse bijections.

We will generalize to a higher direct product. To shorten notation (domain of f_i), we use the recurrence relation $\beta_2(\Lambda_i, W_{i+1}, S_{i+1}, f_i) := \rho(G_{f_i}) = \Lambda_{i+1}$ with initial conditions $\Lambda_1 := W_1$ (see Theorem 7).

Theorem 12. *(Goursat's Theorem for subrings of a direct product of n rings). Let $A_1, A_2, \dots, A_n$ be a finite collection of rings, S is the set of all subrings of $A_1 \times A_2 \times \cdots \times A_n$, and T_{3n-2} is the set of all $(3n-2)$-tuples $Q_{3n-2} := (W_1, W_2, S_2, f_1, W_3, S_3, f_2, \dots, W_n, S_n, f_{n-1})$ where S_j is an ideal of W_j $(j \neq 1)$, W_i is a subring of A_i, and $f_i : \Lambda_i \to \frac{W_{i+1}}{S_{i+1}}$, $(1 \leq i < n)$ is a ring epimorphism. Here sequence $\{\Lambda_i\}_{i=1}^{i=n}$ that satisfies the recurrence relation $\Lambda_{i+1} = \beta_2(\Lambda_i, W_{i+1}, S_{i+1}, f_i) \subseteq A_1 \times A_2 \times \cdots \times A_{i+1}$ with initial conditions $\Lambda_1 := W_1$ and β_2 as defined in Theorem 9. Then there is a one-to-one correspondence S and T_{3n-2}.*

Proof. Define a map $\alpha_n : S \to T_{3n-2}$ by $\alpha_n(R) := (R(1\emptyset), R(2\emptyset), R(2\hat{1}), f_1, R(3\emptyset), R(3\hat{2}), f_2, \dots, R(n\emptyset), R(n\underline{n-1}), f_{n-1}) \in T_{3n-2}$ for all $R \in S$. Conversely, define a map $\beta_n : T_{3n-2} \to S$ by $\beta_n(Q_{3n-2}) := \rho^{-1}(G_{f_{n-1}}) \in S$ for all $Q_{3n-2} = (W_1, W_2, S_2, f_1, W_3, S_3, f_2, \dots, W_n, S_n, f_{n-1}) \in T_{3n-2}$. Now suppose that $R \in S$. Then by Theorem 6, $\beta_n(\alpha_n(R)) = \beta_n(R(1\emptyset), R(2\emptyset), R(2\hat{1}), f_1, R(3\emptyset), R(3\hat{2}), f_2, \dots, R(n\emptyset), R(n\underline{n-1}), f_{n-1}) = \rho^{-1}(G_{f_{n-1}}) = \Lambda_n = \Pi_n(R) = R$. Finally, suppose that $Q_{3n-2} = (W_1, W_2, S_2, f_1, W_3, S_3, f_2, \dots, W_n, S_n, f_{n-1}) \in T_{3n-2}$. Then by Theorem 7, $\alpha_n(\beta_n(Q_{3n-2})) = \alpha_n(\rho^{-1}(G_{f_{n-1}})) = \alpha_n(\Lambda_n) = (\Lambda_n(1\emptyset), \Lambda_n(2\emptyset), \Lambda_n(2\hat{1}), g_1, \Lambda_n(3\emptyset), \Lambda_n(3\hat{2}), g_2, \dots,$

$\Lambda_n(n\emptyset), \Lambda_n(\underline{nn-1}),\ g_{n-1}) = (W_1, W_2, S_2, f_1, W_3, S_3, f_2, \dots, W_n, S_n, f_{n-1}) = Q_{3n-2}$ and this completes the proof.

4. CONCLUSION

The set of all subrings of $A_1 \times A_2 \times \cdots \times A_n$, denoted by S, has a one-to-one correspondence with the set of all $(3n-2)$-tuples $(W_1, W_2, S_2, f_1, W_3, S_3, f_2, \dots, W_n, S_n, f_{n-1})$, denoted by T_{3n-2}, such that S_j is an ideal of $_{W_j}$ $(j \neq 1)$, W_i is a subring of A_i, and $f_i : \Lambda_i \rightarrow \frac{W_{i+1}}{S_{i+1}}(1 \leq i < n)$ is a ring epimorphism, where sequence $\{\Lambda_i\}_{i=1}^{i=n}$ that satisfies the recurrence relation $\Lambda_{i+1} = \beta_2(\Lambda_i, W_{i+1}, S_{i+1}, f_i)$ with initial conditions $\Lambda_1 := W_1$. The mutually inverse maps constructed are $\alpha_n : S \rightarrow T_{3n-2}$ defined by $\alpha_n(R) := (R(1\emptyset), R(2\emptyset), R(2\hat{1}), R(2\hat{1}), f_1,\ R(3\emptyset), R(3\hat{2}), f_2, \dots, R(n\emptyset), R(\underline{nn-1}),\ f_{n-1})$ and $\beta_n : T_{3n-2} \rightarrow S$ defined by $\beta_n(Q_{3n-2}) := \rho^{-1}(G_{f_{n-1}})$.

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