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Parameter Estimation and Hypothesis Testing of Geographically and Temporally Weighted Bivariate Weibull Regression

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Abstract.
In global regression, there is an assumption in the form of an error from a normally distributed model, so data that is normally distributed is required. But in reality, not all of the tested data meet the normal distribution. One of the theoretical distributions of continuous random variables that is often used is the Weibull distribution, where the Weibull distribution is a distribution that is often used to analyze the reliability of an object. If there are two response variables that are correlated with each other, the appropriate method used is Bivariate Weibull Regression (BWR). Spatial data has been widely used in various research fields. The Geographically Weighted Bivariate Weibull Regression (GWBWR) model is a model in which there are spatial effects, where there is spatial heterogeneity in bivariate regression with the response variable being Weibull distribution. In addition, panel data has also been applied in various cases, where panel data can provide information covering more than one time period. This can lead to a temporal effect. This study develops a model that can handle cases of spatial and temporal heterogeneity simultaneously, namely the Geographically and Temporally Weighted Bivariate Weibull Regression (GTWBWR) model. The parameter estimation in the model uses the Maximum Likelihood Estimation (MLE) method which gives results that are not closed-form, so it is continued with the Berndt-Hall-Hall-Hausman (BHHH) numerical iteration.

Keywords: parameter estimation, hypothesis testing, GWBWR

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1. INTRODUCTION

Weibull distribution was originally used in engineering, then later developed in probability theory and statistics. In its development, Weibull distribution has attracted the attention of experts in various fields. Many studies have applied the Weibull distribution [1]. Like the Exponential and Gamma distribution which aims to analyze reliability, but the Weibull distribution is more often used because this distribution is famous for its flexible distribution. Flexibility means that the Weibull distribution can change its distribution into another distribution, such as it can turn into an exponential distribution if it depends on changes in the scale and shape parameters [2].

In linear regression analysis, the relationship between the response variable and the predictor variable is considered constant for each geographic location, so that for each location the parameter estimates are the same. Geographically Weighted Regression (GWR) is a method used to see the relationship between the dependent and independent variables by considering the location (spatial) element. In the regression model, it is generally found that there is a significant effect of the independent variable on the dependent. In the case of GWR, the analysis provides modeling results that are more precise in assessing the relationship between variables in spatial data. The purpose of GWR is to estimate parameters in each research area to determine predictor variables that affect the response variable in each known location [3].

In its development, panel data or temporal data is needed in some cases. This is because panel data has many advantages over cross section data. The advantages of panel data when compared to cross section data are that panel data can provide a larger number of observations, increase degrees of freedom, and reduce collinearity between predictor variables, thereby increasing efficiency [4]. GWR model that applies panel data is the Geographically and Temporally Weighted Regression (GTWR) model, where the model can overcome spatial heterogeneity and collinearity. Many researchers have developed the Weibull regression model into Geographically Weighted Univariate Weibull Regression (GWUWR). There is research that used the GWUWR method which in their research used the Maximum Likelihood Estimation (MLE) method with spatial weighting using Adaptive Gaussian and determining the optimum bandwidth using Generalized Cross Validation (GCV). The test statistic used in this study for simultaneous testing is the Maximum Likelihood Ratio Test (MLRT) and using Wald for partial testing [5]. There is also research that uses two responses with a Weibull distribution, where the resulting model is the Mixed Geographically Weighted Bivariate Weibull Regression (MGWBWR) model. Parameter estimation using the MLE method did not show closed-form results, so the Berndt, Hall, Hall, Hausman (BHHH) optimization method was used [6].

This study develops a model called the Geographically and Temporally Weighted Bivariate Weibull Regression (GTWBWR) model, where the model can handle the spatial and temporal heterogeneity through the development of the Bivariate Weibull Regression (BWR) model with two parameters. The parameter estimation in GTWBWR uses the MLE method, with BHHH iteration and simultaneous parameter testing using MLRT, and partial parameter testing using the Z test statistic for the large sample approach.

2. RESEARCH METHOD

2.1. Bivariate Weibull Regression (BWR)

When BWR is a regression model that has two response variables, namely Y_1 and Y_2, where these variables have a bivariate Weibull distribution with several predictor variables (X). Probability density function on a bivariate Weibull distribution $f(y_1, y_2)$ obtained from the description of the multivariate Weibull distribution function with the following results [7]:

$$f(y_1, y_2) = (\frac{1}{a})^2 (\frac{\sigma_1}{\lambda_1})(\frac{\sigma_2}{\lambda_2})(\frac{y_1}{\lambda_1})^{\frac{\sigma_1}{a}-1} (\frac{y_2}{\lambda_2})^{\frac{\sigma_2}{a}-1}$$

$$exp\{-((\frac{y_1}{\lambda_1})^{\frac{\sigma_1}{a}} + (\frac{y_2}{\lambda_2})^{\frac{\sigma_2}{a}})^a\}[-a(a-1)((\frac{y_1}{\lambda_1})^{\frac{\sigma_1}{a}} + (\frac{y_2}{\lambda_2})^{\frac{\sigma_2}{a}})^{a-2} + a^2((\frac{y_1}{\ddot{e}_1})^{\frac{\sigma_1}{a}} + (\frac{y_2}{\lambda_2})^{\frac{\sigma_2}{a}})^{2a-2}] \quad (1)$$

with the form of the BWR model is:

$$\lambda_k(x) = expexp\ (\beta_k^T x)\ = e^{\beta_k^T x}; k = 1, 2(2)$$

where $\beta_k = [\beta_{k0}\beta_{k1} \dots \beta_{kp}]^T$and $x = [1\ \ x_1 \dots x_p]^T$

2.2. Geographically and Temporally Weighted Bivariate Weibull Regression (GTWBWR)

Spatial and temporal can have an important influence simultaneously on a variable. The GTWR model is known as the development of the GWR model by adding a time dimension. In the GTWR model, it is assumed that the coefficients in the regression have time components and location coordinates [8]. So that the GTWBWR model is a development of the GWBWR model that has been done previously, by providing a time dimension to the estimated regression coefficient parameter. The form of the GTWBWR model equation is:

$$\lambda_{kjl} = expexp\ (\beta_{kl}^T(\upsilon_j)x_{jl})\ = e^{\beta_{kl}^T(\upsilon_j)x_{jl}}; k = 1, 2; j = 1, 2, \dots, n; l = 1, 2, \dots, L(3)$$

Index *j* shows observations in the *j*-th location, index *l* shows the *l*-th period.$x_{jl}$is the observation vector at the *j*-th location of the *l*-th period. $\upsilon_j\ = (u_j, v_j,\ t_j)$is the effect of spatial and temporal. The spatial effect can be seen from u_j, v_jwhich is the coordinates of the *j*-th location and the temporal effect can be seen from$t_j$which is the time of the *j*-th observation. $\beta_{1l}(\upsilon_j)$and $\beta_{2l}(\upsilon_j)$is a regression coefficient parameter vector which has spatial and temporal effects. The GTWBWR model which has the joint probability density function as follows:

$$f(y_{1jl}, y_{2jl}) = (\prod_{k=1}^{2} G_{kjl}) B_{jl} A_{jl}^{a_l - 2} S_{jl},\ j = 1, 2, \ldots, n; l = 1, 2, \ldots, L (4)$$

where,

$$G_{kjl} = \frac{\sigma_{kl}}{a_l} (y_{kjl})^{\frac{\sigma_{kl}}{a_l} - 1} expexp\, [-\frac{\sigma_{kl}}{a_l} \beta_{kl}^T (v_j) x_{jl}]\ ;\ B_{jl} = a_l{}^2 A_{jl}^{a_l} - a_l (a_l - 1)$$

$$A_{jl} = \sum_{k=1}^{2} (y_{kjl})^{\frac{\sigma_{kl}}{a_l}} expexp\, [-\frac{\sigma_{kl}}{a_l} \beta_{kl}^T (v_j) x_{jl}]\ ;\ S_{jl}$$

$$= expexp\, [-(\sum_{k=1}^{2} (y_{kjl})^{\frac{\sigma_{kl}}{a_l}} expexp\, [-\frac{\sigma_{kl}}{a_l} \beta_{kl}^T (v_j) x_{jl}]\ )^{a_l}\ ]$$

2.3. Spatial Temporal Weighting

Spatial-temporal regression requires a weighting that distinguishes it from global regression, where the weighting will vary in each time and location. One of the methods used to measure the distance between location and time is Euclidean distance, where d^S is the spatial distance and d^T is the temporal distance [8].

$$d^{ST} = \varsigma_1 d^S + \varsigma_2 d^T (5)$$

If equation (5) occurs in the *l*-th period, it can be written:

$$(d_{jj*l})^2 = \varsigma_1 [(u_j - u_{j*})^2 + (v_j - v_{j*})^2] + \varsigma_2 (t_j - t_{j*})^2 (6)$$

where$\varsigma_1$is a balancing scale factor for spatial effects and$\varsigma_2$is a scale factor that offsets the temporal effect. This balancing scale factor is useful so that there is no dominance of only one effect. Next, supposeηas parameter ratioς_2/ς_1. If it is determined that$\varsigma_1 = 1$,then obtained$\eta = \varsigma_2$. The value of η can be optimized by cross-validation, so thatR^2or AIC is best. So that the Euclidean distance between location and time is used in determining the spatial-temporal weighting matrix is:

$$d_{jj*l} = \sqrt{(u_j - u_{j*})^2 + (v_j - v_{j*})^2 + \eta (t_j - t_{j*})^2} (7)$$

To form a weighting matrix, it is necessary to define the kernel function first. One of the kernel functions is the adaptive bisquare kernel function, where this function can provide different bandwidths for each observation [3]. The function is:

$$w_{jj*l} = [1 - (\frac{d_{jj*l}}{r_{jl}})^2]^2,\ \ if\ d_{jj*l} < r_{jl}$$

$$= 0 \ \ , \ others (8)$$

$d_{jj*l}$is the distance of spatial-temporal, and$r_{jl}$is bandwidth value at the *j*-th location of the *l*-period One aspect of GTWR is that the estimated parameters depend on the value of the bandwidth used. So in determining the accuracy of the model, it is very dependent on choosing the optimum bandwidth. Generalized Cross-Validation (GCV) is one way that can be used to choose the optimum bandwidth [9].

$$GCV =$$

$$\frac{nL\sum_{l=1}^{L}\ \sum_{j=1}^{n}([y_{1jl}-\hat{y}_{1jl}(r_{1jl})]^{T}[y_{1jl}-\hat{y}_{1jl}(r_{1jl})]+[y_{2jl}-\hat{y}_{2jl}(r_{2jl})]^{T}[y_{2jl}-\hat{y}_{2jl}(r_{2jl})])}{(nL-q_{1})^{2}} (9)$$

where$q_1$is the number of parameters in the model. The minimum GCV will get optimum bandwidth.

2.4. Estimating Parameter and Hypothesis Testing

Parameter estimation in GTWBWR is applied gradually in each period. The first stage is to perform parameter estimation using the first period of the data. The second stage performs parameter estimation using the second period of the data and the previous period, and so on. Up to the *L*-th stage, the parameter estimation is carried out using data from the previous period and data from the *L*-period, where as many as vectors of parameter estimators will be obtained. For each location and time, the parameter β_1 and β_2 will be estimated, so that for each of these parameters a location coordinate and time index will be given. Meanwhile, parameter a, σ_1, and $\sigma_2$will be estimated for each time period only. MLE is an estimation method used by using an adaptive bisquare kernel as a weighting function. Furthermore, MLRT is used for simultaneous hypothesis testing on its parameters.

3. RESULTS AND DISCUSSION

3.1. Parameter Estimation of GTWBWR

Creating the In-likelihood function from the GTWBWR model in equation (4) is the first stage to perform parameter estimation.

$$lnln\ l\ (\beta_{1L}^{T}(v_j)\ \ \beta_{2L}^{T}(v_j)\ \ a_L\ \ \sigma_{1L}\sigma_{2L}; j=1,2,\ldots,n)=\sum_{l=1}^{L}\sum_{j=1}^{n}((\sum_{k=1}^{2} lnln\ G\ _{kjl})$$

$$+lnln\ B_{\ jl} + (a_L - 2)lnln\ A_{\ jl} + lnln\ S_{\ jl})(10)$$

where:

$$G_{kjl} = \frac{\sigma_{kL}}{a_L}(y_{kjl})^{\frac{\sigma_{kL}}{a_L}-1} expexp\left[-\frac{\sigma_{kL}}{a_L}\beta_{kL}^T(v_j)x_{jl}\right] ; A_{jl} = \sum_{k=1}^{2}(y_{kjl})^{\frac{\sigma_{kL}}{a_L}} expexp\left[-\frac{\sigma_{kL}}{a_L}\beta_{kL}^T(v_j)x_{jl}\right]$$

$$B_{jl} = {a_L}^2 A_{jl}^{a_L} - a_L(a_L - 1); S_{jl} = expexp\left[-\left(\sum_{k=1}^{2}(y_{kjl})^{\frac{\sigma_{kL}}{a_L}} expexp\left[-\frac{\sigma_{kL}}{a_L}\beta_{kL}^T(v_j)x_{jl}\right]\right)^{a_L}\right]$$

Estimating the location parameter *j** at time *L* requires a function $Q*_L$, where the function in equation (10) is multiplied by w_{jj*l}, as the spatial-temporal weighting, so we get the following:

$$Q*_L = \sum_{l=1}^{L}\sum_{j=1}^{n} w_{jj*l}\left(\sum_{k=1}^{2} lnln\ G\ *_{kjl} + lnln\ B\ *_{jl} + (a_L - 2)lnln\ A\ *_{jl} + lnln\ S\ *_{jl}\right)(11)$$

where,

$$G*_{kjl} = \frac{\sigma_{kL}}{a_L}(y_{kjl})^{\frac{\sigma_{kL}}{a_L}-1} expexp\left[-\frac{\sigma_{kL}}{a_L}\beta_{kL}^T(v_{j*})x_{jl}\right] ; A*_{jl}$$

$$= \sum_{k=1}^{2}(y_{kjl})^{\frac{\sigma_{kL}}{a_L}} expexp\left[-\frac{\sigma_{kL}}{a_L}\beta_{kL}^T(v_{j*})x_{jl}\right]$$

$$B*_{jl} = {a_L}^2 A_{jl}^{a_L} - a_L(a_L - 1); S*_{jl} = expexp\left[-\left(\sum_{k=1}^{2}(y_{kjl})^{\frac{\sigma_{kL}}{a_L}} expexp\left[-\frac{\sigma_{kL}}{a_L}\beta_{kL}^T(v_{j*})x_{jl}\right]\right)^{a_L}\right]$$

Function$Q*_L$divided into four components for ease of calculation, so that:

$$Q*_L = q_{1L} + q_{2L} + q_{3L} + q_{4L}$$

where:

$$q_{1L} = \sum_{l=1}^{L}\sum_{j=1}^{n} w_{jj*l}\sum_{k=1}^{2} lnln\ G\ *_{kjl}; q_{2L} = \sum_{l=1}^{L}\sum_{j=1}^{n} w_{jj*l}(a_L - 2)lnln\left[\sum_{k=1}^{2}(y_{kjl})^{\frac{\sigma_{kL}}{a_L}} exp\right.$$

$$\left. exp\left[-\frac{\sigma_{kL}}{a_L}\beta_{kL}^T(v_{j*})x_{jl}\right]\right]$$

$$q_{3L} = \sum_{l=1}^{L}\sum_{j=1}^{n} w_{jj*l}lnln\left[{a_L}^2 A_{jl}^{a_L} - a_L(a_L - 1)\right] ; q_{4L} = -\sum_{l=1}^{L}\sum_{j=1}^{n} w_{jj*l}\left(\sum_{k=1}^{2}(y_{kjl})^{\frac{\sigma_{kL}}{a_L}} exp\right.$$

$$exp\ [-\frac{\sigma_{kL}}{a_L}\beta_{kL}^T(v_{j*})x_{jl}]\ )^{a_L}$$

To obtain an estimate of the parameters of the GTWBWR model, it is necessary to maximize equation (11) by finding the first derivative and then equating it with zero.

$$\frac{\partial Q*_L}{\partial\beta_{1L}(v_{j*})}=0;\frac{\partial Q*_2}{\partial\beta_{2L}(v_{j*})}=0;\frac{\partial Q*_L}{\partial a_L}=0;\frac{\partial Q*_L}{\partial\sigma_{1L}}=0;\frac{\partial Q*_L}{\partial\sigma_{2L}}=0$$

The first derivative of the parameters can be seen in Appendix 1.

If the result of the first derivative of equation (11) for each parameter is equalized to zero, it is found that the parameter estimator is not close form, so optimization is needed by using numerical iteration BHHH. The following are the steps of the BHHH algorithm for the GTWBWR model:

1. Initialization$\gamma_{j*L}^{(0)}$and$m=0$with$\psi>0$ for convergence tolerance

$$\hat{\gamma}_{j*L}^{(0)}=[\hat{\beta}_{1L}^{T\ (0)}(v_{j*})\quad\hat{\beta}_{2L}^{T\ (0)}(v_{j*})\hat{a}_L^{(0)}\hat{\sigma}_{1L}^{(0)}\hat{\sigma}_{2L}^{(0)}]^T$$

Mark $\hat{a}_L^{(0)}>0$, $\hat{\sigma}_{1L}^{(0)}>0$, $\hat{\sigma}_{2L}^{(0)}>0$, whereas $\hat{\beta}_{1L}^{T\ (0)}(v_{j*})$ and $\hat{\beta}_{2L}^{T\ (0)}(v_{j*})$is the estimated Weibull regression coefficient.

1. Make vector $g(\hat{\gamma}_{j*L}^{(m)})=[\frac{\partial lnL()}{\partial\beta_{1L}(v_{j*})}\frac{\partial lnL()}{\partial\beta_{2L}(v_{j*})}\frac{\partial lnL()}{\partial a_L}\frac{\partial lnL()}{\partial\sigma_{1L}}\frac{\partial lnL()}{\partial\sigma_{2L}}]^T_{\gamma_{j*L}^{(m)}=\gamma_{j*L}^{(m)}}$

2. Determine the first derivative of density In with respect to parameter

$l_{jl}(\hat{\gamma}_{j*L}^{(m)})^T=[\frac{\partial lnf(y_{1jl},y_{2jl})}{\partial\beta_{1L}(v_{j*})}\frac{\partial lnf(y_{1jl},y_{2jl})}{\partial\beta_{2L}(v_{j*})}\frac{\partial lnf(y_{1jl},y_{2jl})}{\partial a_L}\frac{\partial lnf(y_{1jl},y_{2jl})}{\partial\sigma_{1L}}\frac{\partial lnf(y_{1jl},y_{2jl})}{\partial\sigma_{2L}}]^T_{\gamma_{j*L}^{(m)}=\hat{\gamma}_{j*L}^{(m)}}$, for$j=1,2,\ldots,n$

1. Creating a Hessian matrix$H(\hat{\gamma}_{j*L}^{(m)})=-\sum_{l=1}^{L}\sum_{j=1}^{n}l_{jl}(\hat{\gamma}_{j*L}^{(m)})l_{jl}(\hat{\gamma}_j^{(m)}{}_{j*L})^T$

2. Substitute value$\hat{\gamma}_{j*L}^{(m)}$on element$g(\hat{\gamma}_{j*L}^{(m)})\ and\ H(\hat{\gamma}_{j*L}^{(m)})$

3. Perform iterations starting from m = 0 with $\hat{\gamma}_{j*L}^{(m+1)}=\hat{\gamma}_{j*L}^{(m)}-H^{-1}(\hat{\gamma}_{j*L}^{(m)})g(\hat{\gamma}_{j*L}^{(m)})$

The iteration stops if $\hat{\gamma}_{j*L}^{(m+1)}-\hat{\gamma}_{j*L}^{(m)}<\psi$ when ψ is a very small positive number

1. Repeat from step 2 with m = m + 1 until it converges

2. Optimization is carried out at n locations for each L. So that the result parameter estimator is obtained:

$\hat{\gamma}_{j*L}=[\hat{\beta}_{1L}(v_{j*})\hat{\beta}_{2L}(v_{j*})\hat{a}_L\hat{\sigma}_{1L}\hat{\sigma}_{2L}]^T$, where $L=1,2,\ldots,L*;\ j*=1,2,\ldots,n$

3.2. Hypothesis Testing of GTWBWR

Similar to parameter estimation, hypothesis testing is also carried out in stages in each period. The initial stage in testing the hypothesis is to create a hypothesis for simultaneous testing in period *L*. The hypotheses formed are:

$$H_0 : \beta_{k1L}(v_j) = \beta_{k2L}(v_j) = \cdots = \beta_{kpL}(v_j) = 0 \; ; j = 1, 2, ..., n \; ; k = 1, 2$$

H_1 : minimum there is one $\beta_{khL}(v_j) \neq 0; \; h = 1, 2, \ldots, p$

where *p* is the number of predictor variables in the study.

After determining the hypothesis, it is necessary to maximize the In-likelihood function under the population and under$H_0$to obtain test statistics using the MLRT method. To maximize the function, it is obtained by estimating the parameters using MLE. So first it is necessary to determine the set of parameters under the population and underH_0.

Set of parameters under population:

$$\Omega_L = \{\beta_{1L}(v_j), \beta_{2L}(v_j), a_L, \sigma_{1L}, \sigma_{2L}; j = 1, 2, ..., n\}$$

The set of parameters below:

$$\omega_L = \{\beta_{\omega 1L0}(v_j), \beta_{\omega 2L0}(v_j), a_{\omega L}, \sigma_{\omega 1L}, \sigma_{\omega 2L}; j = 1, 2, ..., n\}$$

The equation (11) is the same as In-likelihood function under the population, and the In-likelihood function under$H_0$is:

$$lnln \; l \; (\omega_L) = lnln \; l \; (\beta_{\omega 1L0}(v_j), \beta_{\omega 2L0}(v_j), a_{\omega L}, \sigma_{\omega 1L}, \sigma_{\omega 2L}; j = 1, 2, ..., n) = \sum_{l=1}^{L} \sum_{j=1}^{n}$$

$$((\sum_{k=1}^{2} lnln \; F_{\;kjl}) + lnln \; D_{\;jl} + (a_{\omega L} - 2) lnln \; C_{\;jl} + lnln \; T_{\;jl})(12)$$

where:

$$F_{kjl} = \frac{\sigma_{\omega kL}}{a_{\omega L}} (y_{kjl})^{\frac{\sigma_{\omega kL}}{a_{\omega L}} - 1} expexp \, [-\frac{\sigma_{\omega kL}}{a_{\omega L}} \beta_{\omega kL0}(v_j)] \; ;$$

$$C_{jl} = \sum_{k=1}^{2} (y_{kjl})^{\frac{\sigma_{\omega kL}}{a_{\omega L}}} expexp \, [-\frac{\sigma_{\omega kL}}{a_{\omega L}} \beta_{\omega kL0}(v_j)]$$

$$D_{jl} = a_{\omega L}{}^2 C_{jl}^{a_{\omega L}} - a_{\omega L}(a_{\omega L} - 1); T_{jl} = expexp \, [-(\sum_{k=1}^{2} (y_{kjl})^{\frac{\sigma_{\omega kL}}{a_{\omega L}}} expexp \, [-\frac{\sigma_{\omega kL}}{a_{\omega L}} \beta_{\omega kL0}(v_j)] \,)^{a_{\omega L}} \;]$$

Next, we need to define the function Q_L^{**}, where the function is a function ln-likelihood under H_0 which is weighted spatial-temporal w_{jj*l}. It is uto get the parameter estimator below H_0 at location *j** period *L*.

$$Q_L^{**} = \sum_{l=1}^{L} \sum_{j=1}^{n} w_{jj*l} (\sum_{k=1}^{2} lnln\ F\ *_{kjl} + lnln\ D\ *_{jl} + (a_L - 2) lnln\ C\ *_{jl} + lnln\ T\ *_{jl}) (13)$$

where:

$$F*_{kjl} = \frac{\sigma_{\omega kL}}{a_{\omega L}} (y_{kjl})^{\frac{\sigma_{\omega kL}}{a_{\omega L}} - 1} exp$$

$$exp\ [-\frac{\sigma_{\omega kL}}{a_{\omega L}} \beta_{\omega kL0}(v_{j*})]\ ; C*_{jl} = \sum_{k=1}^{2} (y_{kjl})^{\frac{\sigma_{\omega kL}}{a_{\omega L}}} expexp\ [-\frac{\sigma_{\omega kL}}{a_{\omega L}} \beta_{\omega kL0}(v_{j*})]$$

$$D*_{jl} = a_{\omega L}{}^2 C_{jl}^{a_{\omega L}} - a_{\omega L}(a_{\omega L} - 1); T*_{jl} = expexp\ [-(\sum_{k=1}^{2} (y_{kjl})^{\frac{\sigma_{\omega kL}}{a_{\omega L}}} exp$$

$$exp\ [-\frac{\sigma_{\omega kL}}{a_{\omega L}} \beta_{\omega kL0}(v_{j*})]\ )^{a_{\omega L}}]$$

The next step is to create the first derivative of equation (13) for each parameter below H_0 then equated to zero. Function Q_L^{**} divided into 4 components, so that:

$$Q_L^{**} = q *_{1L} + q *_{2L} + q *_{3L} + q*_{4L}$$

where:

$$q*_{1L} = \sum_{l=1}^{L} \sum_{j=1}^{n} w_{jj*l} \sum_{k=1}^{2} lnln\ F\ *_{kjl}; q*_{2L} = \sum_{l=1}^{L} \sum_{j=1}^{n} w_{jj*l} (a_{\omega L} - 2) lnln\ [C*_{jl}]$$

$$q*_{3L} = \sum_{l=1}^{L} \sum_{j=1}^{n} w_{jj*l} lnln\ [a_{\omega L}{}^2 (C*_{jl})^{a_{\omega L}} - a_{\omega L}(a_{\omega L} - 1)]\ ; q*_{4L} = -\sum_{l=1}^{L} \sum_{j=1}^{n} w_{jj*l} (C*_{jl})^{a_{\omega L}}$$

The derivative of the function Q_L^{**} against the parameters can be seen in Appendix 2.

If the first derivative of the function Q_L^{**} to each of the parameters below H_0 equated with zero results in a parameter estimator that is not closed-form, so optimization is needed by using numerical iteration with BHHH method, which has been described previously so that the parameter estimator vector below is obtained: H_0

$\hat{\gamma}_{\omega j*L} = [\hat{\beta}_{\omega 1L0}(v_{j*})\ \hat{\beta}_{\omega 2L0}(v_{j*})\ \ \hat{a}_{\omega L}\ \hat{\sigma}_{\omega 1L}\ \hat{\sigma}_{\omega 1L}\ ]^T$, where j*=1,2,...,n

To facilitate decision making, the following test statistics are used:

$$G_L^2 = -lnln\ \square_L^2\ = -lnln\ (\frac{L(\hat{\omega}_L)}{L(\hat{\square}_L)})^2 = 2[lnln\ L\ (\hat{\square}_L) - lnln\ L\ (\hat{\omega}_L)] \sim_{nL\to\infty} \chi^2_{(\alpha,2npL)} (14)$$

$L(\hat{\Omega}_L)$ is the likelihood function with the parameter estimator under the population and $L(\hat{\omega}_L)$ is the likelihood function with the parameter estimator below H_0. Statistics G_L^2 can be approximated by a chi-square distribution with 2*npL* degrees of freedom if the sample size is large, where 2*npL* is the difference between the parameters under H_0 and the parameters under the population. Area of rejection H_0 that is $G_L^2 > \chi^2_{(\alpha;2npL)}$ [10]. If the results are rejected H_0 in the form of simultaneous testing, it is necessary to do partial testing. The partial test has the hypothesis as follows:

$$H_0 : \beta_{khL}(v_j) = 0 H_1 : \beta_{khL}(v_j) \neq 0$$

where *k* = 1,2; *h* = 1,2,...,*p*; *L* = 1,2,...,*L**

The test statistics used are:

$$Z = \frac{\hat{\beta}_{khL}(v_j)}{se(\hat{\beta}_{khL}(v_j))} \sim_{nL\to\infty} N(0,1) (15)$$

with $se(\hat{\beta}_{khL}(v_j)) = \sqrt{v\hat{a}r(\hat{\beta}_{khL}(v_j))}$, for $v\hat{a}r(\hat{\beta}_{khL}(v_j))$ obtained from the *k*+1 diagonal of the matrix $v\hat{a}r(\hat{\gamma}_{j*L}) = -H^{-1}(\hat{\gamma}_{j*L})$.

The Z statistic will approach the standard normal distribution if the sample size is large, so the rejection region H_0 is $Z > Z_{\alpha/2}$ [11].

4. CONCLUSION

The estimation of parameter in the GTWBWR model using the MLE method produces an estimator that is not closed-form, so optimization is needed by using numerical iterations of BHHH at each time period. To test the hypothesis simultaneously using the MLRT method with the G^2 test statistic with a chi-square distribution and the partial test using the Z test statistic which is normally distributed.

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APPENDIX 1.

Derivative$Q*_L$against parameter$\beta_{kL}(v_{j*})$:

$$\frac{\partial Q*_L}{\partial \beta_{kL}(v_{j*})} = \frac{\partial q_{1L}}{\partial \beta_{kL}(v_{j*})} + \frac{\partial q_{2L}}{\partial \beta_{kL}(v_{j*})} + \frac{\partial q_{3L}}{\partial \beta_{kL}(v_{j*})} + \frac{\partial q_{4L}}{\partial \beta_{kL}(v_{j*})}$$

where:

$$\frac{\partial q_{1L}}{\partial \beta_{kL}(v_{j*})} = \sum_{l=1}^{L} \sum_{j=1}^{n} (-\frac{\sigma_{kL}}{a_L} w_{jj*l} x_{jl})$$

$$\frac{\partial q_{2L}}{\partial \beta_{kL}(v_{j*})} = \sum_{l=1}^{L} \sum_{j=1}^{n} w_{jj*l}(a_L - 2)\frac{-\sigma_{kL} x_{jl}}{a_L} A*_{jl}{}^{-1}(y_{kjl})^{\frac{\sigma_{kL}}{a_L}} exp$$

$$exp\,[-\frac{\sigma_{kL}}{a_L}\beta_{kL}^T(v_{j*})x_{jl}]$$

$$\frac{\partial q_{3L}}{\partial \boldsymbol{\beta}_{kL}\left(\mathbf{v}_{j^*}\right)} = \sum_{l=1}^{L}\sum_{j=1}^{n}\frac{w_{jj^*l}}{a_L{}^2 A_{jl}^{*\,a_L} - a_L\left(a_L-1\right)}\left[-a_L{}^2 A_{jl}^{*\,a_L-1}\left(y_{kjl}\right)^{\frac{\sigma_{kL}}{a_L}} \boldsymbol{\sigma}_{kL}\mathbf{x}_{j1}\exp\left[-\frac{\sigma_{kL}}{a_L}\boldsymbol{\beta}_{kL}{}^{\mathrm{T}}\left(\mathbf{v}_{j^*}\right)\mathbf{x}_{j1}\right]\right]$$

$$\frac{\partial q_{4L}}{\partial \beta_{kL}(v_{j*})} = \sum_{l=1}^{L} \sum_{j=1}^{n} w_{jj*l} A*_{jl}{}^{a_L-1}\sigma_{kL} x_{jl}(y_{kjl})^{\frac{\sigma_{kL}}{a_L}} expexp\,[-\frac{\sigma_{kL}}{a_L}\beta_{kL}^T(v_{j*})x_{jl}]$$

Derivative$Q*_L$against parametera_L

$$\frac{\partial Q*_L}{\partial a_L} = \frac{\partial q_{1L}}{\partial a_L} + \frac{\partial q_{2L}}{\partial a_L} + \frac{\partial q_{3L}}{\partial a_L} + \frac{\partial q_{4L}}{\partial a_L}$$

where:

$$\frac{\partial q_{1L}}{\partial a_L} = -\sum_{l=1}^{L}\sum_{j=1}^{n}\frac{2w_{ii^*l}}{a_L} - \frac{1}{a_L{}^2}\sum_{l=1}^{L}\sum_{j=1}^{n}\sum_{k=1}^{2}\sigma_{kL} w_{jj^*l}\left(\ln\left(y_{kjl}\right) - \boldsymbol{\beta}_{kL}^T\left(\mathbf{v}_{j^*}\right)\mathbf{x}_{j1}\right)$$

$$\frac{\partial q_{2L}}{\partial a_L} = \sum_{l=1}^{L} \sum_{j=1}^{n} w_{jj*l}[lnln\ A\ *_{jl} + (a_L - 2)\frac{A*_{jl}{}^{-1}}{a_L{}^2}\sum_{k=1}^{2}\sigma_{kL}(y_{kjl})^{\frac{\sigma_{kL}}{a_L}} exp$$

$$exp\,[-\frac{\sigma_{kL}}{a_L}\beta_{kL}^T(v_{j*})x_{il}]\ (\beta_{kL}{}^T(v_{j*})x_{jl} - lnln\ y\ _{kjl})]$$

$$\frac{\partial q_{3L}}{\partial a_L} = \sum_{l=1}^{L}\sum_{j=1}^{n}\frac{w_{jj^*l}}{a_L{}^2 A_{jl}^{*\,a_L} - a_L\left(a_L-1\right)}\left[2a_L A_{jl}^{*\,a_L} + a_L{}^2 A_{jl}^{*\,a_L}\left[\ln A_{jl}^* + \frac{A_{jl}^{*\,-1}}{a_L}\sum_{k=1}^{2}\sigma_{kL}\left(y_{kjl}\right)^{\frac{\sigma_{kL}}{a_L}}\exp\left[-\frac{\sigma_{kL}}{a_L}\boldsymbol{\beta}_{kL}^T\left(\mathbf{v}_{j^*}\right)\mathbf{x}_{i1}\right]\left(\boldsymbol{\beta}_{kL}^T\left(\mathbf{v}_{j^*}\right)\mathbf{x}_{j1} - \ln y_{kjl}\right)\right] - 2a_L + 1\right]$$

$$\frac{\partial q_{4L}}{\partial a_L} = -\sum_{l=1}^{L} \sum_{j=1}^{n} w_{jj*l} A*_{jl}{}^{a_L}[lnln\ A\ *_{jl} + \frac{A*_{jl}{}^{-1}}{a_L}\sum_{k=1}^{2}\sigma_{kL}(y_{kjl})^{\frac{\sigma_{kL}}{a_L}} exp$$

$$exp\,[-\frac{\sigma_{kL}}{a_L}\beta_{kL}^T(v_{j*})x_{jl}]\ (\beta_{kL}^T(v_{j*})x_{jl} - lnln\ y\ _{kjl})]$$

Derivative$Q*_L$against parameterσ_{kL}:

$$\frac{\partial Q*_L}{\partial \sigma_{kL}} = \frac{\partial q_{1L}}{\partial \sigma_{kL}} + \frac{\partial q_{2L}}{\partial \sigma_{kL}} + \frac{\partial q_{3L}}{\partial \sigma_{kL}} + \frac{\partial q_{4L}}{\partial \sigma_{kL}}$$

where:

$$\frac{\partial q_{1L}}{\partial \sigma_{kL}} = \sum_{l=1}^{L}\sum_{j=1}^{n}\frac{w_{jj^*l}}{\sigma_{kL}} + \sum_{l=1}^{L}\sum_{j=1}^{n}\frac{w_{jj^*l}}{a_L}\left(\ln y_{kjl} - \boldsymbol{\beta}_{kL}^T\left(\mathbf{v}_{j^*}\right)\mathbf{x}_{j1}\right) \quad \frac{\partial q_{2L}}{\partial \sigma_{kL}} = \sum_{l=1}^{L} \sum_{j=1}^{n} w_{jj*l}(a_L - 2)\frac{A*_{jl}{}^{-1}}{a_L}(y_{kjl})^{\frac{\sigma_{kL}}{a_L}} exp$$

$$exp\ [-\frac{\sigma_{kL}}{a_L}\beta^T_{kL}(v_{j*})x_{jl}]\ (lnln\ y\ _{kjl}-\beta^T_{kL}(v_{j*})x_{jl})$$

$$\frac{\partial q_{3L}}{\partial \sigma_{kL}}=\sum_{l=1}^{L}\sum_{j=1}^{n}\frac{w_{ii^*l}}{a_L{}^2A^{*\,a_L}_{jl}-a_L\left(a_L-1\right)}\left[a_L{}^2A^{*\,a_L-1}_{jl}\left(y_{kjl}\right)^{\frac{\sigma_{kL}}{a_L}}\exp\left[-\frac{\sigma_{kL}}{a_L}\boldsymbol{\beta}^T_{kL}\left(\mathbf{v}_{j^*}\right)\mathbf{x}_{jl}\right]\left(\ln y_{kjl}-\boldsymbol{\beta}^T_{kL}\left(\mathbf{v}_{j^*}\right)\mathbf{x}_{jl}\right)\right]$$

$$\frac{\partial q_{4L}}{\partial \sigma_{kL}}=-\sum_{l=1}^{L}\sum_{j=1}^{n}w_{jj*l}A*_{jl}{}^{a_L-1}(y_{kjl})^{\frac{\sigma_{kL}}{a_L}}exp$$

$$exp\ [-\frac{\sigma_{kL}}{a_L}\beta^T_{kL}(v_{j*})x_{il}]\ (lnln\ y\ _{kjl}-\beta^T_{kL}(v_{j*})x_{jl})$$

APPENDIX 2.

Q^{**}_Lagainst parameter$\beta_{\omega kL0}(v_{j*})$:

$$\frac{\partial Q^{**}_L}{\partial\beta_{\omega kL0}(v_{j*})}=\frac{\partial q*_{1L}}{\partial\beta_{\omega kL0}(v_{j*})}+\frac{\partial q*_{2L}}{\partial\beta_{\omega kL0}(v_{j*})}+\frac{\partial q*_{3L}}{\partial\beta_{\omega kL0}(v_{j*})}+\frac{\partial q*_{4L}}{\partial\beta_{\omega kL0}(v_{j*})}$$

where:

$$\frac{\partial q*_{1L}}{\partial\beta_{\omega kL0}(v_{j*})}=\sum_{l=1}^{L}\sum_{j=1}^{n}(-\frac{\sigma_{kL}}{a_L}w_{jj*l})$$

$$\frac{\partial q*_{2L}}{\partial\beta_{\omega kL0}(v_{j*})}=\sum_{l=1}^{L}\sum_{j=1}^{n}w_{jj*l}(a_{\omega L}-2)\frac{-\sigma_{\omega kL}}{a_{\omega L}}C*_{jl}{}^{-1}(y_{kjl})^{\frac{\sigma_{\omega kL}}{a_{\omega L}}}expexp\ [-\frac{\sigma_{\omega kL}}{a_{\omega L}}\beta_{\omega kL0}(v_{j*})]$$

$$\frac{\partial q^*_{3L}}{\partial\beta_{\omega kL0}\left(\mathbf{v}_{j^*}\right)}=\sum_{l=1}^{L}\sum_{j=1}^{n}\frac{w_{jj^*l}}{a_{\omega L}{}^2C^{*\,a_{\omega L}}_{jl}-a_{\omega L}\left\{a_{\omega L}-1\right\}}\left[-a_{\omega L}{}^2C^{*\,a_{\omega L}-1}_{jl}\left(y_{kjl}\right)^{\frac{\sigma_{\omega kL}}{a_{\omega L}}}\sigma_{\omega kL}\exp\left[-\frac{\sigma_{\omega kL}}{a_{\omega L}}\beta_{\omega kL0}\left(\mathbf{v}_{j^*}\right)\right]\right]$$

$$\frac{\partial q*_{4L}}{\partial\beta_{\omega kL0}(v_{j*})}=\sum_{l=1}^{L}\sum_{j=1}^{n}w_{jj*l}C*_{jl}{}^{a_{\omega L}-1}\sigma_{\omega kL}(y_{kjl})^{\frac{\sigma_{\omega kL}}{a_{\omega L}}}expexp\ [-\frac{\sigma_{\omega kL}}{a_{\omega L}}\beta_{\omega kL0}(v_{j*})]$$

DerivativeQ^{**}_Lagainst parameter$a_{\omega L}$:

$$\frac{\partial Q^{**}_L}{\partial a_{\omega L}}=\frac{\partial q*_{1L}}{\partial a_{\omega L}}+\frac{\partial q*_{2L}}{\partial a_{\omega L}}+\frac{\partial q*_{3L}}{\partial a_{\omega L}}+\frac{\partial q*_{4L}}{\partial a_{\omega L}}$$

where:

$$\frac{\partial q^*_{1L}}{\partial a_{\omega L}}=-\sum_{l=1}^{L}\sum_{j=1}^{n}\frac{2w_{jj^*l}}{a_{\omega L}}-\frac{1}{a_{\omega L}{}^2}\sum_{l=1}^{L}\sum_{j=1}^{n}\sum_{k=1}^{2}\sigma_{\omega kL}w_{jj^*l}\left(\ln\left(y_{kjl}\right)-\beta_{\omega kL0}\left(\mathbf{v}_{j^*}\right)\right)$$

$$\frac{\partial q*_{2L}}{\partial a_{\omega L}}=\sum_{l=1}^{L}\sum_{j=1}^{n}w_{jj*l}[lnln\ C\ *_{jl}+(a_{\omega L}-2)\frac{C*_{jl}{}^{-1}}{a_{\omega L}{}^2}\sum_{k=1}^{2}\sigma_{\omega kL}(y_{kjl})^{\frac{\sigma_{\omega kL}}{a_{\omega L}}}exp$$

$$exp\ [-\frac{\sigma_{\omega kL}}{a_{\omega L}}\beta_{\omega kL0}(\upsilon_{j*})]\ (\beta_{\omega kL0}(\upsilon_{j*}) - lnln\ y\ _{kjl})]$$

$$\frac{\partial q_{3L}^{*}}{\partial a_{\omega L}} = \sum_{l=1}^{L}\sum_{j=1}^{n}\frac{w_{jj^{*}l}}{a_{\omega L}{}^{2}C_{jl}^{*\,a_{\omega L}} - a_{\omega L}\left(a_{\omega L}-1\right)}\left[2a_{\omega L}C_{jl}^{*\,a_{\omega L}} + a_{\omega L}{}^{2}C_{jl}^{*\,a_{\omega L}}\left[\ln C_{jl}^{*} + \frac{C_{jl}^{*\,-1}}{a_{\omega L}}\sum_{k=1}^{2}\sigma_{\omega kL}\left(y_{kjl}\right)^{\frac{\sigma_{\omega kL}}{a_{\omega L}}}\exp\left[-\frac{\sigma_{\omega kL}}{a_{\omega L}}\beta_{\omega kL0}\left(\mathbf{v}_{j^{*}}\right)\right]\left(\beta_{\omega kL0}\left(\mathbf{v}_{j^{*}}\right) - \ln y_{kjl}\right)\right] - 2a_{\omega L} + 1\right]$$

$$\frac{\partial q*_{4L}}{\partial a_{\omega L}} = -\sum_{l=1}^{L}\sum_{j=1}^{n} w_{jj*l}C*_{jl}{}^{a_{\omega L}}[lnln\ C\ \ *_{jl} + \frac{C*_{jl}{}^{-1}}{a_{\omega L}}\sum_{k=1}^{2}\sigma_{\omega kL}(y_{kjl})^{\frac{\sigma_{\omega kL}}{a_{\omega L}}}\ exp$$

$$exp\ [-\frac{\sigma_{\omega kL}}{a_{\omega L}}\beta_{\omega kL0}(\upsilon_{j*})]\ (\beta_{\omega kL0}(\upsilon_{j*}) - lnln\ y\ _{kjl})]$$

DerivativeQ_{L}^{**} against parameter $\sigma_{\omega kL}$:

$$\frac{\partial Q_{L}^{**}}{\partial \sigma_{\omega kL}} = \frac{\partial q*_{1L}}{\partial \sigma_{\omega kL}} + \frac{\partial q*_{2L}}{\partial \sigma_{\omega kL}} + \frac{\partial q*_{3L}}{\partial \sigma_{\omega kL}} + \frac{\partial q*_{4L}}{\partial \sigma_{\omega kL}}$$

where:

$$\frac{\partial q_{1L}^{*}}{\partial \sigma_{\omega kL}} = \sum_{l=1}^{L}\sum_{j=1}^{n}\frac{w_{jj^{*}l}}{\sigma_{\omega kL}} + \sum_{l=1}^{L}\sum_{j=1}^{n}\frac{w_{jj^{*}l}}{a_{\omega L}}\left(\ln y_{kjl} - \beta_{\omega kL0}\left(\mathbf{v}_{j^{*}}\right)\right)$$

$$\frac{\partial q*_{2L}}{\partial \sigma_{\omega kL}} = \sum_{l=1}^{L}\sum_{j=1}^{n} w_{jj*l}(a_{\omega L} - 2)\frac{C*_{jl}{}^{-1}}{a_{\omega L}}(y_{kjl})^{\frac{\sigma_{\omega kL}}{a_{\omega L}}}\ exp$$

$$exp\ [-\frac{\sigma_{\omega kL}}{a_{\omega L}}\beta_{\omega kL0}(\upsilon_{j*})]\ (lnln\ y\ _{kjl} - \beta_{\omega kL0}(\upsilon_{j*}))$$

$$\frac{\partial q_{3L}^{*}}{\partial \sigma_{\omega kL}} = \sum_{l=1}^{L}\sum_{j=1}^{n}\frac{w_{jj^{*}l}}{a_{\omega L}{}^{2}C_{jl}^{*\,a_{\omega L}} - a_{\omega L}\left(a_{\omega L}-1\right)}\left[a_{\omega L}{}^{2}C_{jl}^{*\,a_{\omega L}-1}\left(y_{kjl}\right)^{\frac{\sigma_{\omega kL}}{a_{\omega L}}}\exp\left[-\frac{\sigma_{\omega kL}}{a_{\omega L}}\beta_{\omega kL0}\left(\mathbf{v}_{j^{*}}\right)\right]\left(\ln y_{kjl} - \beta_{\omega kL0}\left(\mathbf{v}_{j^{*}}\right)\right)\right]$$

$$\frac{\partial q*_{4L}}{\partial \sigma_{\omega kL}} = -\sum_{l=1}^{L}\sum_{j=1}^{n} w_{jj*l}C*_{jl}{}^{a_{\omega L}-1}(y_{kjl})^{\frac{\sigma_{\omega kL}}{a_{\omega L}}}\ exp$$

$$exp\ [-\frac{\sigma_{\omega kL}}{a_{\omega L}}\beta_{\omega kL0}(\upsilon_{j*})]\ (lnln\ y\ _{kjl} - \beta_{\omega kL0}(\upsilon_{j*}))$$

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