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Parameter Estimation and Hypothesis Testing of Geographically and Temporally Weighted Bivariate Negative Binomial Regression

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Abstract.
When the response variable is discrete as a number (count) and there is a violation of the assumption of equidispersion, namely overdispersion or underdispersion then one of the appropriate alternative models used is Negative Binomial Regression (NBR). Moreover, if there are two correlated response variables and have an equidispersion violation, the Bivariate Negative Binomial Regression (BNBR) model is the solution. However, the BNBR model is considered inappropriate if the data contains spatial and temporal heterogeneity derived from panel data with the unit of observation in the form of a region. Therefore, a model is offered which is known as Geographically and Temporally Weighted Bivariate Negative Binomial Regression (GTWBNBR) which accommodates spatial and temporal effects. This study aims to conduct parameter estimates and test statistics for the GTWBNBR model. Estimated parameters use Maximum Likelihood Estimation (MLE) with BHHH numerical iteration because the MLE estimates are not closed-form. When the sample size is large, the Maximum Likelihood Ratio Test (MLRT) is used for simultaneous parameter testing while the test statistic for partial parameter testing approaches the Chi-Square distribution so that it can be tested using the Z-Test.

Keywords: parameter estimation, hypothesis testing, GTWBNBR

1. INTRODUCTION

The proper regression model for the variable response as count number is the Poisson regression model [1]. Poisson regression modeling has assumptions that must be fulfilled, in which the response variable's mean is the same as the variance (equidispersion) [2]. If there is a violation in assumption then the probability that occurs is that the average value is smaller than variance (overdispersion) or bigger than variance (underdispersion) [3]. The effect of the violation provides underestimation of standard error and the conclusions about the significance test become inappropriate [4]. Frequently, violation of equidispersion is overdispersion, then one of the analysis methods to conquer it is Negative Binomial Regression [2, 5–8]. A global model will be obtained with Negative Binomial Regression analysis, which is the model that applies to all regions in which the data is retrieved. There are differences in geography between regions which illustrate the effects of spatial heterogeneity between regions. The method of analysis that accommodates the presence of spatial heterogeneity is known as Geographically Weighted Regression (GWR) [9].

The parameter estimation in GWR method using Weighted Least Squares (WLS) and optimum bandwidth selection with cross validation (CV) function [9, 10]. The gaussian adaptive kernel functions and the best models using Akaike's Information Criterion (AIC) and corrected AIC (AICc) functions have been used [10]. Research with spatial approaches to discrete data that is overdispersed has been widely done. Mostly the estimation method used is MLE with Newton Raphson iteration (NR) and parameter testing with MLRT method [11–13]. Alternatives to the NR iteration are the Nelder Mead iteration [14] and the Berndt-Hall-Hall-Hausman iteration (BHHH) [15].

Recently, many problems require spatial analysis involving time-period or temporal effects. The GWR spatial analysis method which is accommodating temporal effects known as Geographically and Temporally Weighted Regression (GTWR) method [15, 16]. GTWR modeling involves the observation in the previous period to do modeling in a period, thus affecting the effectiveness and effectiveness compared to the GWR model. This study will use spatial temporal analysis methods for discrete response variables that are overdispersed using GTWBNBR analysis.

2. RESEARCH METHOD

2.1. Bivariate Negative Binomial Regression (BNBR)

When a pair of random discrete count variables Y_1 and Y_2 has correlation and overdispersion with dispersion paramater τ, then BNBR is considered. BNBR distribution defined as in the following equation [17].

$$p\left(y_{1i}, y_{2i}\right) = G\mu_{1i}^{\ y_{1i}} \mu_{2i}^{\ y_{2i}} \tau^{-\tau^{-1}} \left(\tau^{-1} + \mu_{1i} + \mu_{2i}\right)^{-\left(\tau^{-1} + y_{1i} + y_{2i}\right)} \quad (1)$$

where $G = \dfrac{\Gamma\left(\tau^{-1} + y_{1i} + y_{2i}\right)}{\Gamma(\tau^{-1})\Gamma(y_{1i} + 1)\Gamma(y_{2i} + 1)}$; $y_{1i} = 0, 1, 2, \ldots$; $y_{2i} = 0, 1, 2, \ldots$; $i = 1, 2, \ldots, n$ with

$$E\left(Y_{ci}\right) = \mu_{ci}; c = 1, 2; Var\left(Y_{ci}\right) = \mu_{ci}\left(1 + \tau\mu_{ci}\right); Corr(Y_{1i}, Y_{2i}) = \sqrt{\frac{\mu_{1i}\mu_{2i}\tau^{2}}{\left(1 + \tau\mu_{1i}\right)\left(1 + \tau\mu_{2i}\right)}}.$$

The BNBR model for the *i*-th observation of n samples is

$$\mu_{ci} = \mathrm{e}^{\left(\beta_{c0}+x_{1i}\beta_{c1}+\ldots+x_{pi}\beta_{cp}\right)}; i = 1, 2, \ldots, n; c = 1, 2; \quad (2)$$

MLE method is used to estimate the parameter combined with BHHH numerical iteration while parameter testing uses MLRT.

2.2. Geographically and Temporally Weighted Bivariate Negative Binomial Regression (GTWBNBR)

The GTWBNBR model is an advanced BNBR model which accommodates spatial temporal effects. Refers to equation (1), then the join pmf of Y_{1il} and Y_{2il} of GTWBNBR as follows:

$$\left(Y_{1il}, Y_{2il}\right) \sim BNB\left(\mu_{1il}\left(s_i\right), \mu_{2il}\left(s_i\right), \tau_l\right) \quad (3)$$

$$p\left(y_{1il}, y_{2il}\right) = G_l \mu_{1il}\left(s_i\right)^{y_{1il}} \mu_{2il}\left(s_i\right)^{y_{2il}} \tau_l^{-\tau_l^{-1}} \left(\tau_l^{-1} + \mu_{1il} + \mu_{2il}\right)^{-\left(\tau_l^{-1}+y_{1il}+y_{2il}\right)},$$

where $G_l = \dfrac{\Gamma\left(\tau_l^{-1} + y_{1il} + y_{2il}\right)}{\Gamma(\tau_l^{-1})\Gamma(y_{1il}+1)\Gamma(y_{2il}+1)}$ then the model is $\mu_{1il} = \mathrm{e}^{\left(\boldsymbol{\beta}_{il}^{T}{}_{1l}(s_i)\right)}$ and $\mu_{2il} = e^{\left(\boldsymbol{\beta}_{il}^{T}{}_{2l}(s_i)\right)}$

l-index is the period of the total period of L while *i*-index is the *i*-th observed from n random sample, so the total observation is $n \times L$. x_{il} it is a vector-sized$(p + 1) \times 1$ at every *i*-th location and at every *l*-th period. Furthermore $s_i = \left(u_i, v_i, t_i\right)$ is the *i*-th observed coordinates spatial temporal, while u_i is the location of the latitude, v_i is the location of the longitude coordinate points of the *i*-th observation and t_i is period of the *i*-th observation. $\boldsymbol{\beta}_{1l}\left(s_i\right)$ and $\boldsymbol{\beta}_{2l}\left(s_i\right)$ is a parameter vector of the regression coefficient with a temporal spatial effect sized $(p + 1) \times 1$.

2.3. Spatial Temporal Weighting

Basically, location and period in GTWR modeling are measured in different units. Let spatial distance is d^S and temporal distance is d^T. Then the spatial-temporal distance is [18]

$$d^{ST} = \gamma d^S + \eta d^T \quad (4)$$

In balancing the spatial-temporal effect, a scale factor is required which notated with γ and , thus equation (4) become

$$\left(d_{ii^*}^{ST}\right)^2 = \gamma\left[\left(u_i - u_{i^*}\right)^2 + \left(v_i - v_{i^*}\right)^2\right] + \eta\left(t_i - t_{i^*}\right)^2 \quad (5)$$

let $\delta = \frac{\eta}{\gamma}$, then spatial-temporal distance is $\frac{\left(d_{ii^*}^{ST}\right)^2}{\gamma} = \left[\left(u_i - u_{i^*}\right)^2 + \left(v_i - v_{i^*}\right)^2\right] + \delta\left(t_i - t_{i^*}\right)^2$.The value of γ assumed 1 for ease calculation in δ. δ used to increase or decrease temporal distances to match spatial distances. The value of δ can be optimized by using the smallest AIC value. Thus, the spatial- temporal euclidean distance can be written[18]

$$d_{ii^*l} = \sqrt{\left(u_i - u_{i^*}\right)^2 + \left(v_i - v_{i^*}\right)^2 + \delta_l\left(t_i - t_{i^*}\right)^2} \quad (6)$$

When building a weighting matrix, a kernel function must be determined. There are two types of the kernel, adaptive and fixed. Through this adaptive approach allows every observation to have a different bandwidth [19]. The adaptive bi-square kernel functions are shown as follows:

$$w_{ii^*l} = \begin{cases} \left(1 - \left(\frac{d_{ii^*l}}{q_{il}}\right)^2\right)^2 ; & for\ d_{ii^*l} \le q_{il} \\ 0 & \end{cases} \quad (7)$$

where d_{ii^*l} is spasial-temporal distance and q_{il} is bandwidth. The accuracy of parameter estimation depends on the bandwidth selection, the more optimum the more accurate [19]. Methods to determine optimal bandwidth are (Cross Validation) and GCV (Generalized Cross Validation) [20]. GCV is preferred because it is quite easy to work with and provides better precision in parameter estimation [19, 20]. The GCV method is defined as follows:

$$GCV = \frac{nL\sum_{l=1}^{L}\sum_{i=1}^{n}\left[\left(\left[\mathbf{y}_{1il} - \hat{\mathbf{y}}_{1il}(q_{1il})\right]^T\left[\mathbf{y}_{1il} - \hat{\mathbf{y}}_{1il}(q_{1il})\right]\right) + \left(\left[\mathbf{y}_{2il} - \hat{\mathbf{y}}_{2il}(q_{2il})\right]^T\left[\mathbf{y}_{2il} - \hat{\mathbf{y}}_{2il}(q_{2il})\right]\right)\right]}{\left(nL - a_1\right)^2} \quad (8)$$

a_1 states the number of parameters in the model. The minimum CGV, the more optimum bandwidth.

2.4. Estimating Parameter and Hypothesis Testing

The data that is used in GTWBNBR modeling is panel data. The parameters testing in GTWBNBR is applied gradually in every period which the previous period incorporate into the next period. For every location and period, the parameter β_{1L} and $\beta_{2L}$is estimated, so every parameter is given an index indicating the location and period, while period index is only given to the overdispersion paramete τ which means that parameters only estimated for every period. MLE method with BHHH iterations applied in estimating parameters as well as adaptive bi-square kernel functions. The stages in the estimation using MLE are:

1. Define the In-likelihood function of the GTWBNBR model
2. Multiply (i) by spatial-temporal weight (Q_l^*)
3. Derivating (ii) against every parameter then equaled to zero to maximize function
4. When the result of stage (iii) is not closed-form, then an iteration is considered. The iteration used in this study is

The BHHH iteration because it has the advantage of only requiring the first derivative to build the Hessian matrix.

Furthermore, the stages in obtaining test statistics on GTWBNBR are as follows:

1. Define a hypothesis to test the GTWBNBR model
2. Specify the set of parameters under H_0 (ω_l) and the set of parameters underpopulation for every period $(\square_l)$
3. Define In-likelihood function stage (b) and multiply it by spatial-temporal weight ($Q_{\omega l}^*$)
4. Maximizing stage (c) by derivating against every parameter then equaled to zero. When the result of the derivation is not closed form, then BHHH iteration is considered

Determining odds ratio to get statistic test, the distribution and critical rejection area.

3. RESULTS AND DISCUSSION

3.1. Parameter Estimation of GTWBNBR

The first stage in estimating parameters is defining ln-likelihood function that refers to equation (3) with the following equation form as follows:

$$
\begin{aligned}
&L\{\boldsymbol{\beta}_{1L}(s_i),\boldsymbol{\beta}_{2L}(s_i),\tau_L; i=1,2,\dots,n; l=1,2,\dots,L\} = \left[\prod_{i=1}^{n} p(y_{1i1},y_{2i1})\right]\left[\prod_{i=1}^{n} p(y_{1i2},y_{2i2})\right]\dots\left[\prod_{i=1}^{n} p(y_{1iL},y_{2iL})\right] \\
&= \prod_{l=1}^{L}\prod_{i=1}^{n}\left[\prod_{k=1}^{y_{1i1}+y_{2i1}}\left(y_{1il}+y_{2il}+\tau_l^{-1}-k\right)\left(\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{1l}(s_i)\right)\right)^{y_{1il}}\left(\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{2l}(s_i)\right)\right)^{y_{2il}}\tau_l^{-\tau_l^{-1}}\right. \\
&\left.\left(\tau_l^{-1}+\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{1l}(s_i)\right)+\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{2l}(s_i)\right)\right)^{-\left(\tau_l^{-1}+y_{1il}+y_{2il}\right)}\right]
\end{aligned}
\quad (9)
$$

with $x_{il} = [1 x_{1il} x_{2il} \cdots x_{pil}]^T$

Estimated parameters for every location i^* at any period that has been given temporal spatial weight w_{ii^*l} namely Q_l^* with the following equation form as follows:

$$
\begin{aligned}
Q_l^* &= \sum_{l=1}^{L}\sum_{i=1}^{n} w_{ii^*l} \ln\prod_{i=1}^{n} p(y_{1il},y_{2il}) \\
&= \sum_{l=1}^{L}\sum_{i=1}^{n} w_{ii^*l}\left[\sum_{k=1}^{y_{1i1}+y_{2i1}} \ln\left(y_{1il}+y_{2il}+\tau_l^{-1}-k\right) + y_{1il}\mathbf{x}_{il}^T\boldsymbol{\beta}_{1l}(s_{i^*}) + y_{2il}\mathbf{x}_{il}^T\boldsymbol{\beta}_{2l}(s_{i^*}) - \frac{\ln\tau_l}{\tau_l} - \left(y_{1il}+y_{2il}+\tau_l^{-1}\right)\right. \\
&\left.\ln\left(\tau_l^{-1}+\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{1l}(s_{i^*})\right)+\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{1l}(s_{i^*})\right)\right) - \ln(y_{1il}!) - \ln(y_{2il}!)\right]
\end{aligned}
\quad (10)
$$

Furthermore, to get the estimated parameters of the GWTBNBR model at every observation point (i^*-th location) in every l-period, then the equation (10) derived against every of its parameters and equated to zero (detailed drop attached).

The first derivation of Q_l^* against $\boldsymbol{\beta}_{1il}(s_{i^*})$

$$
\frac{\partial Q_l^*}{\partial\boldsymbol{\beta}_{1l}(s_{i^*})} = \sum_{l=1}^{L}\sum_{i=1}^{n} w_{ii^*l}\left[\frac{\left(y_{1il}-\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{1l}(s_{i^*})\right)\right)\mathbf{x}_{il}^T}{\left(1+\tau_l\left(\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{1l}(s_{i^*})\right)+\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{2l}(s_{i^*})\right)\right)\right)}\right] = 0 \quad (11)
$$

The first derivation of Q_l^* against $\boldsymbol{\beta}_{2il}(s_{i^*})$

$$
\frac{\partial Q_l^*}{\partial\boldsymbol{\beta}_{2l}(s_{i^*})} = \sum_{l=1}^{L}\sum_{i=1}^{n} w_{ii^*l}\left[\frac{\left(y_{2il}-\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{2l}(s_{i^*})\right)\right)\mathbf{x}_{il}^T}{\left(1+\tau_l\left(\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{1l}(s_{i^*})\right)+\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{2l}(s_{i^*})\right)\right)\right)}\right] = 0 \quad (12)
$$

The first derivation of Q_l^* against τ_l

$$\frac{\partial Q_l^*}{\partial \tau_l} = \sum_{l=1}^{L}\sum_{i=1}^{n} w_{ii^*l}\left(\sum_{k=1}^{y_{1il}+y_{2il}}\left(\frac{y_{1il}+y_{2il}-k}{\left(1+\tau_l y_{1il}+\tau_l y_{2il}-\tau_l k\right)}\right)+\frac{\ln\left(1+\tau_l\left(\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{1l}\left(s_{i^*}\right)\right)+\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{2l}\left(s_{i^*}\right)\right)\right)\right)}{\tau_l^2} - \frac{\left(\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{1l}\left(s_{i^*}\right)\right)+\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{2l}\left(s_{i^*}\right)\right)\right)}{\tau_l}\right) \quad (13)$$

The first derivation of Q_l^* against every parameter when equated to zero, it is not closed-form, then BHHH iteration is considered. The stages of the BHHH algorithm for the GTWBNBR model are provided earlier.

3.2. Hypothesis Testing of GTWBNBR

Parameter testing is applied gradually in every period. Parameter testing consists of partial and simultaneous testing. The hypothesis-testing parameters simultaneously as follows:

$$H_0 : \beta_{c1L}(s_i) = \beta_{c2L}(s_i) = \cdots = \beta_{cpL}(s_i) = 0; c = 1,2; i = 1,2,\ldots,n$$

H_1 : There is at least one $\beta_{cjL}(s_i) \neq 0; c = 1,2; j = 1,2,\ldots,p$

where p represents the number of predictor variables.

Testing parameters simultaneously in GTWBNBR model using the MLRT method. The first stage is determining the set of parameters under $H_0(\omega_L)$ and the set of parameters underpopulation (□$_L$), as follow:

Set of parameters under H_0 (ω_L):

$$\omega_L = \{\beta_{\omega 10L}(s_i), \beta_{\omega 20L}(s_i)\tau_{\omega L}; i = 1,2,\ldots,n\}(14)$$

Set of parameters underpopulation:

□$_L = \{\beta_{1L}(s_i), \beta_{2L}(s_i), \tau_L; i = 1,2,\ldots,n\}(15)$

The likelihood function underpopulation ($L($□$_L)$) refers to the equation (9), while the In-likelihood function under H_0 $ln(L(\omega_L))$ as follows :

$$\ln\left(L\left(\omega_L\right)\right) = \sum_{l=1}^{L}\sum_{i=1}^{n}\left[\sum_{k=1}^{y_{1il}+y_{2il}}\left(y_{1il}+y_{2il}+\tau_{\omega L}^{-1}-k\right)+y_{1il}\mathbf{x}_{il}^T\boldsymbol{\beta}_{\omega 10L}\left(s_i\right)+y_{2il}\mathbf{x}_{il}^T\boldsymbol{\beta}_{\omega 20L}\left(s_i\right)-\frac{\ln\tau_{\omega L}}{\tau_{\omega L}}-\left(y_{1il}+y_{2il}+\tau_{\omega L}^{-1}\right)\ln\left(\tau_{\omega L}^{-1}+\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{\omega 10L}\left(s_i\right)\right)+\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{\omega 20L}\left(s_i\right)\right)\right)-\ln\left(y_{1il}!\right)-\ln\left(y_{2il}!\right)\right] \quad (16)$$

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Furthermore, determine the function $Q^*_{\omega L}$ i.e. In likelihood under H_0 in equation (16) multiplied by temporal spatial weighting $w_{ii^*l}$to obtain parameter estimators under H_0

$$
\begin{aligned}
Q^*_{\omega L} &= \sum_{l=1}^{L}\sum_{i=1}^{n} w_{ii^*l} \ln L\left\{\boldsymbol{\beta}_{\omega 10L}\left(s_{i^*}\right), \boldsymbol{\beta}_{\omega 20L}\left(s_{i^*}\right), \tau_{\omega L}\right\} \\
&= \sum_{l=1}^{L}\sum_{i=1}^{n} w_{ii^*l}\left[\sum_{k=1}^{y_{1il}+y_{2il}} \ln\left(y_{1il}+y_{2il}+\tau_{\omega L}^{-1}-k\right) + y_{1il}\mathbf{x}_{il}^T\boldsymbol{\beta}_{\omega 10L}\left(s_{i^*}\right) + y_{2il}\mathbf{x}_{il}^T\boldsymbol{\beta}_{\omega 20L}\left(s_{i^*}\right) - \frac{\ln \tau_{\omega L}}{\tau_{\omega L}}\right. \\
&\quad \left. -\left(y_{1il}+y_{2il}+\tau_{\omega L}^{-1}\right)\ln\left(\tau_{\omega L}^{-1}+\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{\omega 10L}\left(s_{i^*}\right)\right)+\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{\omega 10L}\left(s_{i^*}\right)\right)\right) - \ln\left(y_{1il}!\right) - \ln\left(y_{2il}!\right)\right]
\end{aligned} \quad (17)
$$

Then look for the first derivative $Q^*_{\omega L}$ against every parameter under the H_0 and then equated with zero and obtained the following result:

The first derivation of $Q^*_{\omega L}$ against $\beta_{\omega 10L}(s_{i^*})$ is

$$
\frac{\partial Q^*_{\omega L}}{\partial \beta_{\omega 10L}\left(s_{i^*}\right)} = \sum_{l=1}^{L}\sum_{i=1}^{n} w_{ii^*l}\left[\frac{\left(y_{1il}-\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{\omega 10L}\left(s_{i^*}\right)\right)\right)\mathbf{x}_{il}^T}{\left(1+\tau_{\omega L}\left(\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{\omega 10L}\left(s_{i^*}\right)\right)+\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{\omega 20L}\left(s_{i^*}\right)\right)\right)\right)}\right] \quad (18)
$$

The first derivation of $Q^*_{\omega L}$ against $\beta_{\omega 20L}(s_{i^*})$ is

$$
\frac{\partial Q^*_{\omega L}}{\partial \beta_{\omega 20L}\left(s_{i^*}\right)} = \sum_{l=1}^{L}\sum_{i=1}^{n} w_{ii^*l}\left[\frac{\left(y_{2il}-\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{\omega 20L}\left(u_{i^*},v_{i^*},t_{i^*}\right)\right)\right)\mathbf{x}_{il}^T}{\left(1+\tau_{\omega L}\left(\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{\omega 10L}\left(u_{i^*},v_{i^*},t_{i^*}\right)\right)+\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{\omega 20L}\left(u_{i^*},v_{i^*},t_{i^*}\right)\right)\right)\right)}\right] \quad (19)
$$

The first derivation of $Q^*_{\omega L}$ against $\tau_{\omega L}$ is

$$
\begin{aligned}
\frac{\partial Q^*_{\omega L}}{\partial \tau_{\omega L}} = \sum_{l=1}^{L}\sum_{i=1}^{n} w_{ii^*l}&\left(\sum_{k=1}^{y_{1il}+y_{2il}}\left(\frac{y_{1il}+y_{2il}-k}{\left(1+\tau_{\omega L}y_{1il}+\tau_{\omega L}y_{2il}-\tau_{\omega L}k\right)}\right)+\frac{\ln\left(1+\tau_{\omega L}\left(\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{\omega 10L}\left(s_{i^*}\right)\right)+\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{\omega 20L}\left(s_{i^*}\right)\right)\right)\right)}{\tau_{\omega L}^2}\right. \\
&\left. -\frac{\left(\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{\omega 10L}\left(s_{i^*}\right)\right)+\exp\left(\mathbf{x}_{il}^T\boldsymbol{\beta}_{\omega 20L}\left(s_{i^*}\right)\right)\right)}{\tau_{\omega L}}\right)
\end{aligned} \quad (20)
$$

The equation obtained from the first derivative in (19) to (21) is not closed-form, then be solved using the BHHH iteration. Thus obtained the estimated parameters under H_0 as follows:

$$
\hat{\boldsymbol{\theta}}_{\omega iL}(s_{i^*}) = [\hat{\boldsymbol{\beta}}^T_{\omega 10L}(s_{i^*})\hat{\boldsymbol{\beta}}^T_{\omega 20L}(u_{i^*}, v_{i^*}, t_{i^*})\hat{\tau}_{\omega L}]^T \quad (21)
$$

where $i = 1, 2, \dots, n$ and $L = 1, 2, \dots, L^*$

The results of the estimating parameters underpopulation and H_0 with MLE will maximize the likelihood function in every L-th period, as follows :

$$\ln L(\hat{\omega}_L) = \ln L\left(\left(\hat{\beta}_{\omega 10L}(s_i), \hat{\beta}_{\omega 20L}(s_i), \hat{\tau}_{\omega L}\right); i = 1,2,...,n; L = 1,2,...,L^*\right)$$
$$= \sum_{l=1}^{L}\sum_{i=1}^{n} w_{ii^*l}\left[\sum_{k=1}^{y_{1il}+y_{2il}} \ln\left(y_{1il} + y_{2il} + \tau_{\omega L}^{-1} - k\right) + y_{1il}\boldsymbol{\beta}_{il}^{T}{}_{\omega 10L}(s) + y_{2il}\boldsymbol{\beta}_{il}^{T}{}_{\omega 20L}(s_i)\right.$$
$$\left. - \frac{\ln \tau_{\omega L}}{\tau_{\omega L}} - \left(y_{1il} + y_{2il} + \tau_{\omega L}^{-1}\right)\ln\left(\tau_{\omega L}^{-1} + \exp\left(\boldsymbol{\beta}_{il}^{T}{}_{\omega 10L}(s_i)\right) + \exp\left(\boldsymbol{\beta}_{il}^{T}{}_{\omega 20L}(s_i)\right)\right) - \ln(y_{1il}!) - \ln(y_{2il}!)\right] \quad (22)$$

$$\ln L(\hat{\Omega}_L) = \ln L\left(\left(\hat{\boldsymbol{\beta}}_{1L}(s_i), \hat{\boldsymbol{\beta}}_{2L}(s_i), \hat{\tau}_L\right); i = 1,2,...,n; L = 1,2,...,L^*\right)$$
$$= \sum_{l=1}^{L}\sum_{i=1}^{n} w_{ii^*l}\left[\sum_{k=1}^{y_{1il}+y_{2il}} \ln\left(y_{1il} + y_{2il} + \tau_L^{-1} - k\right) + y_{1il}\boldsymbol{\beta}_{il}^{T}{}_{1L}(s_i) + y_{2il}\boldsymbol{\beta}_{il}^{T}{}_{2L}(s_i)\right.$$
$$\left. - \frac{\ln \tau_L}{\tau_L} - \left(y_{1il} + y_{2il} + \tau_L^{-1}\right)\ln\left(\tau_L^{-1} + \exp\left(\boldsymbol{\beta}_{il}^{T}{}_{1L}(s_i)\right) + \exp\left(\boldsymbol{\beta}_{il}^{T}{}_{2L}(s_i)\right)\right) - \ln(y_{1il}!) - \ln(y_{2il}!)\right] \quad (23)$$

Based on the equation (22) and (23) further determined odds ratio as follows:

$$\Lambda_L = \frac{L(\hat{\omega}_L)}{L(\hat{\Omega}_L)} < \Lambda_0 (24)$$

Reject H_0 if $\Lambda_L < \Lambda_{L0}$, where $0 \leq \Lambda_{L0} \leq 1$ with $\alpha = P(\Lambda_L < \Lambda_{L0} H otrue)$. Λ_{L0} It is a constant that depends on α. Furthermore equation (24) can be written as follows:

$$G_L^2 = -ln\Lambda^2{}_L = -lnln\left(\frac{L(\hat{\omega}_L)}{L(\hat{\Omega}_L)}\right)^2 = 2(lnlnL(\hat{\Omega}_L) - lnlnL(\hat{\omega}_L)) \sim_{nL\to\infty} \chi^2_{(\alpha;a-b)} (25)$$

$G_L^2$is approach with χ^2 distribution with dof a-b, where a is the total of parameters under population and b is the total parameters under H_0. Reject H_0 if $G_L^2 > \chi^2_{(\alpha;a-b)}$. If the results of the test simultaneously reject H_0, then proceed with a partial test to find out whether there is an influence of predictor variables on responses individually and how big the effect is. The hypothesis in the testing parameter partially as follows:

$$H_0 : \beta_{cjL}(s_i) = 0$$

$$H_1 : \beta_{cjL}(s_i) \neq 0 (26)$$

This test is done for $c = 1, 2; j = 1, 2, \dots, p; i = 1, 2, \dots, n$. So that the test statistics used:

$$Z = \frac{\hat{\beta}_{cjL}(s_i)}{SE\left(\hat{\beta}_{cjL}(s_i)\right)} \sim_{nL\to\infty} N(0,1) \quad (27)$$

where $se(\hat{\beta}_{cjL}(s_i)) = \sqrt{v\hat{a}r(\hat{\beta}_{cjL}(s_i))}$ and $v\hat{a}r(\hat{\beta}_{cjL}(s_i))$ obtained from diagonal element of varian kovarian matrix $v\hat{a}r(\hat{\theta}_{iL}) = -H^{-1}(\hat{\theta}_{iL})$. In big size sample, statistik Z will approach normal standard distribution, with rejection area H_0 is $Z > Z_{\frac{\alpha}{2}}$.

4. CONCLUSION

Estimating the parameters of the GTWBNBR model using the MLE method does not produce a closed form so that it requires numerical iteration optimization. BHHH as a numerical iteration is applied to every period. Parameter testing consists of simultaneous and partial testing which is applied to every period using the MLRT method. When the sample is large, the test statistic on the simultaneous parameter test is approximated by the Chi-Square distribution, while the partial parameter test is approximated by the Normal Standard distribution.

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