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Optimal Fuel Saving in 4D Waypoint Networks

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Abstract

The purpose of this work is to develop a trajectory optimization method that generates a fuel optimal trajectory from a predefined 4D waypoint networks, where the arrival time is specified for each waypoint in the network. A single source shortest path algorithm is presented to generate the optimal flight trajectory that minimizes fuel burn. Generating such trajectories enables the airlines to cope with increasing fuel costs and to reduce aviation induced climate change, as emission is directly related to the amount of fuel burn. Two case studies were considered and the simulation results showed that flying a fuel optimal trajectory based on the proposed algorithm leads to a reduction of average fuel consumption on international flights by 2-4% compared with the conventional trip fuel.

Keywords: Fuel saving, Cost index, 4D trajectory optimization, Waypoint network, Dijkstra's algorithm

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1. Introduction

Improving aircraft operational efficiency has become a dominant topic in air transportation, as the airlines around the world have seen the price of fuel has risen sharply during the last decades. Currently, air transportation accounts for about 2% of total global CO_2 emissions and about 12% of the CO_2 from all transportation source [1]. The increased fuel prices and environmental concerns have pushed airlines to reduce fuel consumption and to find margins for performance improvements. Efforts to modernize the aircraft fleet are limited by an extremely slow and expensive process of new aircraft adoption, which can take decades, therefore it is important to find different alternatives to reduce the fuel consumption in current aircraft, which will likely to share the sky with most modern aircraft in near future. One of these alternatives is to optimize flight trajectories and traffic control procedure. The existing flight planning techniques are suboptimal. Hence, a fuel optimal flight path can significantly save fuel.

A practical solution that reduces the cost associated with time and fuel consumption during flight is the Cost Index (*CI*). The value of the *CI* reflects the relative effects of fuel cost on overall trip cost as compared to time-related direct operating cost. For all aircraft models, the minimum value of cost index equal to zero results in maximum range airspeed and minimum trip fuel, but this configuration ignores the time cost. If the cost index is maximum, the flight time is minimum, the velocity and the Mach number are maximum, but ignores the fuel cost [2]. In this study, the Cost Index assumes to be zero as only fuel cost is taken into consideration.

$$CI = \frac{TimeCost \sim (€/hr)}{FuelCost \sim (€/kg)} \quad (1)$$

Recent studies propose that, during the take-off and climb phase of the flight, accelerating and flap retraction at a lower altitude than the typical 3000 ft decrease the fuel consumption, lower flap setting cause low drag, resulting less fuel burn during climb, it also suggest that descending at a higher slope angle than 30 enable the aircraft to save fuel [3], [4]. By improving the cruise speed and altitude profiles is possible to reduce fuel burn in cruise phase, Hagelauer and Mora-Camino [3] conducted a study based on a constant value of Cost Index for a given arrival time, in order to find the optimum cruise speed and altitude profile. An alternative way to conserve fuel in current aircraft is by flying optimal trajectories. The trajectory optimization problem can be solved by various kinds of methods, however, these methods can be classified into two basic approaches: the indirect approach and the direct approach [5], [6].

The trajectory optimization problem is solved by the pontryagin maximum principle [7] in the indirect approach, where the original optimal control problem is converted into Eular- Lagrange system (boundary value problem) by formulating the first-order necessary condition which derived from pontryagin maximum principle. Generally, the indirect approach leads to more accurate results than the direct approach. However, in general, a rather good initial approximation of the co-state equation is required in order to convergence, which is quite difficult to guess as the physical meaning of co-estate equations are not well established [8]. Besides for many practical optimal control problems, these boundary values problems are quite difficult to solve, because of complex dynamics and constraints structure, which results in two-point boundary value problem (TPBVPs), it demands computationally intensive iterative numerical procedures.

On the other hand, the direct approach is based on the transformation of optimal control problem into a parameter optimization problem [9]. Which is done by discretizing the infinite-dimensional problem into a finite-dimensional problem and later on solving it by the nonlinear programming. Direct methods tend to have better convergence properties over indirect methods. Another great advantage of direct methods is that they do not have to deal with the co-state equation. The parameterization techniques have an important role in the convergence and accuracy of the solution. The most known direct approaches are based on Runge-Kutta scheme [10] and collocation methods [11]. Recently, some works have been presented for higher nonlinear dynamic system called a Chebyshev pseudo-spectral method [12], [13], [14]. That procedure is based on the approximation of both controls and state by interpolating polynomials at the Chebyshev nodes. However experimental results show that the approximation of controls by higher-order polynomials give rise to excessive wavy curves for the states.

Recently Some research activities have been done for 4D optimal trajectory generation. Bousson and Gameiro [15] presented a quintic spline approach for 4D trajectory generation for UAVs. Boukraa, Bestaoui and Azouz [16] proposed a 3D optimal trim trajectories planner algorithm to generate trajectories for a set of predefined waypoints in space. Ahmed and Bousson [17] generated a time-optimal trajectory from 4D predefined networks.

In this present paper, applying shortest path algorithms in graph theory, an optimal trajectory has been approximated by the path that minimizes the total link cost connecting the origin and destination in a pre-defined network. The graph methods often require large computation time and memory space but guarantee global optimal solutions. In this paper, the single source shortest path algorithm was used to generate the fuel optimal trajectory.

This study is restricted to the climb, cruise and descent phases of the flight and ignores the take-off and landing approach, and assuming the initial and final waypoints are at an altitude of 3000 feet, where, in the initial waypoint the aircraft begins the climb phase and in the final waypoint the aircraft begins the landing approach. This work primarily attempts to quantify benefits of fuel optimal trajectory which was found by implying the Dijkstra's shortest path algorithm. In this work, a benefit is meant to imply a reduction in fuel burn due to using the Dijkstra's shortest path algorithm to the actual unimproved flight.

2. Problem Formulation

The main goal of this paper is to find a fuel optimal path from a predefined 4D waypoint networks. A representation of waypoint networks is shown in Figure 1, where P_1 is the initial waypoint and P_N is the final waypoint of the networks.

Most of the approaches consider the waypoints defined by tri-dimensional coordinate positions. $P_k = (\lambda_k, \varphi_k, h_k)^T$ where, $k = 1, 2, ..., i, j, ..., N$ and do not take into account the time. By adding the arrival time restriction to the tri-dimensional waypoint it is possible to define the 4D waypoints as $P_k = (\lambda_k, \varphi_k, h_k, \tau_k)^T$. Where, $\lambda_k, \varphi_k, h_k, \tau_k$ are respectively longitude, latitude, altitude and arrival time at waypoint P_k.

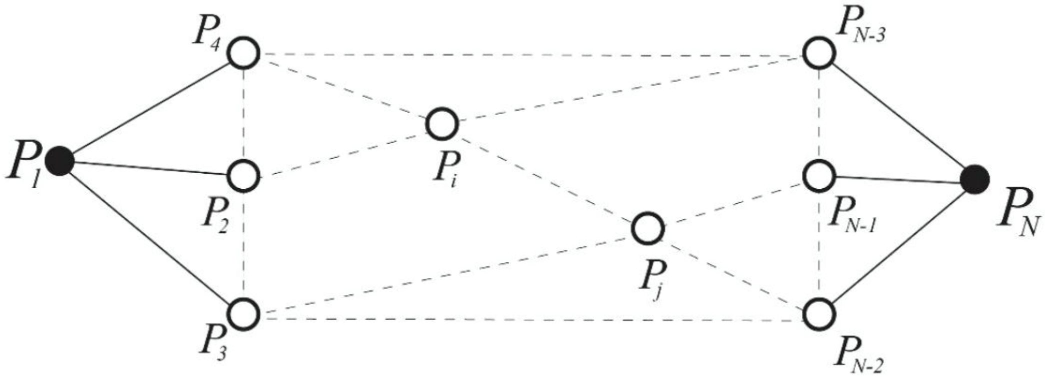

Figure 1: Representation of 4D waypoint networks

As trajectory generation requires a geocentric coordinates system, the 4D waypoints need to be transformed from the accustomed geodetic coordinate system to geocentric coordinates. Now to transform the geodetic coordinates the following equations need to be applied [18].

$$x_j = (N_j + h_j) \cos \varphi_j \cos \lambda_j \tag{2}$$

$$y_j = (N_j + h_j) \cos \varphi_j \sin \lambda_j \tag{3}$$

$$z_j = [N_j(1 - e^2) + h_j] \sin \varphi_j \tag{4}$$

Being a the Earth semi-major axis and e its eccentricity, N_j can be calculated as follows:

$$N_j = \frac{a}{\sqrt{1 - e^2 \sin^2 \varphi_j}} \tag{5}$$

Now the 4D waypoints can be demonstrated in geocentric coordinates as follows:

$$P_j = (x_j, y_j, z_j, \tau_j)^T \tag{6}$$

The problem to be solved is to navigate the aircraft along with 4D waypoints as in Eq. (6) starts from the initial waypoint P_1 to the final waypoint P_N such that it minimizes the total fuel consumption by the aircraft. The performance index to be minimized in this problem can be written in the integral form as:

$$J = \int_{\tau_0}^{\tau_f} (f + CI^* \tau) dt \tag{7}$$

Where, f and τ represent the fuel burn and flight time of the full trajectory from waypoint P_1 to waypoint P_N. CI is the cost index as in Eq. (1), it is an adjustable parameter which is chosen by the airlines to balance the fuel and time costs. In this problem, the Cost Index assumes to be zero as only fuel cost is taken into consideration.

The following section proposes a method that will determine the fuel optimal path along with specified waypoints from a 4D waypoint network by implying the Dijkstra's shortest path algorithm.

3. Proposed Method

To generate a fuel optimal trajectory from a set of waypoints in 4D waypoint network requires finding the associated fuel consumed the other, defined as: df_k by the aircraft to go from one waypoint to

$$df_k = f_{nom} \times d\tau_k \tag{8}$$

Where, f_{nom} [kg/min] is the nominal fuel flow rate, df_k [kg] is the amount of fuel consumed and $d\tau_k$ is the amount of time needed by the aircraft to go from waypoints P_{k-1} to P_k and, which can be described in the following equations:

$$df_k = f_k - f_{k-1} \tag{9}$$

$$d\tau_k = \tau_k - \tau_{k-1} \tag{10}$$

Where, f_k [kg] and τ_k [min] are respectively the fuel burn and flight time required to get to waypoint P_k from initial waypoint. The nominal fuel flow rate f_{nom}, can be estimated by the thrust and thrust specific fuel consumption as follows:

$$f_{nom} = \eta \times Thr \tag{11}$$

However, the f_{nom} varies with specific aircraft and with different flight phases, as the thrust in Eq.(11) is different in different phases of flight. The Base of Aircraft Data (BADA) model provides coefficients that allow to calculate the thrust specific fuel consumption η and different thrust level Thr, which can be used to calculate the f_{nom} in different phases of the flight [19], [20].

4. Dijkstra's Algorithm

Dijkstra's algorithm, was first proposed by the Dutch computer scientist Edsger Dijkstra in 1956 and published in 1959, is the most well-known shortest path algorithm. This is a graph search algorithm that solves the single-source shortest path problem for a graph with non- negative edge path costs, producing a shortest-path tree. The most common variant of the algorithm fixes one vertex as the source and another as the destination vertex and find the shortest path between them.

Dijkstra's algorithm solves the single-source shortest-paths problem on a weighted, directed graph G (V, E) where V is a set of vertices and E is a set of edges on the graph. This algorithm requires 3 variables as input in order to find the path with the lowest cost between the source and destination vertices, they are respectively the graph, the source vertex, and the destination vertex, and at the end, it returns a reduced graph as output.

This algorithm will determine the global optimal (best route to take), given a number of vertices and edges as long as it has the graph as an input, no matter how large the graph is. In addition to the basic formulation of Dijkstra's algorithm, the following aspects must be defined specifically for the flight trajectory optimization problem. The number of vertices V, the edges E between the vertices and the source and destination vertices. In this paper, the waypoints of the 4D waypoint networks are the vertices V, the initial waypoint is the source vertex s, the final waypoint is the destination vertex and the associated travel time $d\tau_k$ by the aircraft between the pairs of waypoints are the edges E between these vertices (waypoints).

In Figure 2 a full execution of the Dijkstra's shortest path algorithm operation is shown. The circles represent the vertices or nodes and the lines with arrows are the edges. Each edge has a non-negative cost associated with it. The problem is to find the most cost-efficient route from the source vertex to any other vertex.

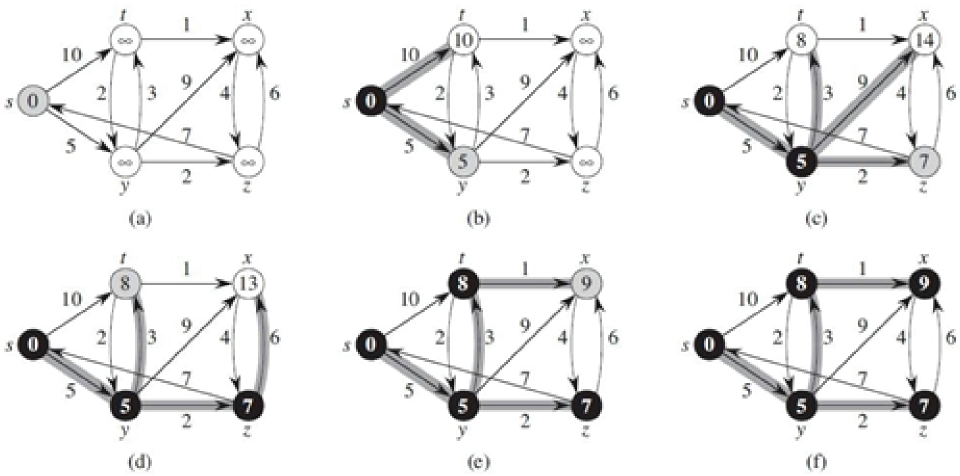

Figure 2: The execution of Dijkstra's algorithm

In this example, the source vertex s is the leftmost vertex. The value with low-cost estimates appear within the vertices, and shaded edges indicate predecessor values. Black vertices are already examined thus they have the value of the lowest cost associated with them to go from the source vertex, and the white vertices are going to be examined. The first step (a) shows the situation just before the first iteration of the while loop. Form step (b) to step (f) shows the situation after each successive iteration of the while loop. The value of lowest cost and predecessors shown in last step (f), and these are the final values of the lowest cost to go to that vertex from the source vertex [21–23].

5. Modeling of 4D Waypoints Network

The following differential equations are the dynamic model used to model the problem:

$$\dot{x} = V \cos\gamma \cos\psi \quad (12)$$

$$\dot{y} = V \cos\gamma \sin\psi \quad (13)$$

$$\dot{z} = V \sin\gamma \quad (14)$$

$$\dot{V} = u_1 \quad (15)$$

$$\dot{\gamma} = u_2 \quad (16)$$

$$\dot{\psi} = u_3 \quad (17)$$

where, (x, y, z) are the geocentric coordinate system, the V ,γ, and ψ are the velocity, flight path angle, and heading respectively, the variables u_1, u_2, and u_3 are respectively the acceleration, the flight path angle rate, and the heading rate. The state and control vectors are composed by $X = [x, y, z, V, \gamma, \psi]$ and $U = [u_1, u_2, u_3]$ respectively. Considering the following constraints: Due to aerodynamic, structural and propulsive limitations, bound constraints are imposed on the state and control vectors as follow:

$$V^{\min} \le V \le V^{\max} \quad (18)$$

$$\gamma^{\min} \le \gamma \le \gamma^{\max} \quad (19)$$

$$\psi^{\min} \le \psi \le \psi^{\max} \quad (20)$$

$$u_i^{\min} \le u_i \le u_i^{\max},\ i = 1, 2, 3 \quad (21)$$

6. Simulation and Result

In this section, the simulation and result of the fuel optimal trajectories are presented for two different case studies. In the first example, a short-haul flight Lisbon to Geneva and in the second example a medium-haul flight Lisbon to Stockholm were considered. In both examples, the fuel optimal trajectories were generated by using Dijkstra's algorithm. All the analysis of the simulation has been done using Matlab 2016[a].

Example 1

This subsection presents the simulation and results of example 1 where a short-haul flight, Lisbon to Geneva was considered. The 4D waypoint network of this short-haul flight consists of two trajectories, and has total of 22 waypoints including the initial and final waypoints, and each trajectory has 12 waypoints including the initial and final waypoints.

TABLE 1: List of waypoints in 1st trajectory for short-haul flight

waypoint	$x[m]$	$y[m]$	$z[m]$	$d\tau_k$ [min]	df_k [kg]
Initial (P1)	2647.235288	-421.1992558	2155.785264	0	0
P2	2644.86006	-414.5681541	2161.780769	2.371751	291.0612
P3	2639.688421	-400.3464116	2173.301325	3.437191	348.0156
P4	2626.183424	-359.68012	2199.831101	7.295251	535.3803
P5	2617.384267	-321.3434642	2217.611029	5.748997	294.6361
P6	2513.665622	-148.0478476	2351.006175	32.48854	1176.085
P7	2455.327051	5.585254511	2415.971579	23.71969	858.6528
P8	2440.964861	153.4156637	2425.576695	19.97808	723.2065
P9	2432.450512	164.6407432	2431.936166	2.058735	7.823191
P10	2415.587867	198.5338065	2443.073738	5.71202	32.05871
P11	2404.139607	224.5489423	2449.734117	5.275779	42.20624
Final (P22)	2389.738702	240.8587387	2460.555523	5.618537	54.83692
Total				113.7046	4363.963

Boeing 737-700 (B737) aircraft was used to analyze the flight trajectories. (table 1 and 2) Show the waypoints lists for both of the trajectories. Each waypoint is defined in geocentric coordinates (x, y, z), the travel time $d\tau_k$ and consumed fuel df_k between the waypoints are also shown. To find the fuel optimal trajectory from the 4D waypoint network possible connection between waypoints in both trajectories were established,

and their travel time $d\tau_k$ and consumed fuel df_k between these possible waypoints connections were calculated.

Table 2: List of waypoints in 2nd trajectory for short-haul flight

waypoint	$x[m]$	$y[m]$	$z[m]$	$d\tau_k$ [min]	df_k [kg]
Initial (P1)	2647.235288	-421.1992558	2155.785264	0	0
P12	2646.148436	-411.3007568	2160.834393	2.862798	351.3225
P13	2641.765637	-394.8066635	2171.800352	3.66612	371.1947
P14	2631.351084	-350.7310454	2195.127395	7.374258	541.1783
P15	2624.264915	-308.9768094	2211.270337	6.036433	309.3672
P16	2581.499623	-79.54841825	2280.219078	32.6647	1182.462
P17	2485.884636	24.1455534	2384.614747	23.55432	852.6664
P18	2445.165301	155.0221788	2421.268939	19.04446	689.4095
P19	2437.263718	170.4015841	2426.749665	2.415762	9.179894
P20	2421.774537	205.9872305	2436.367639	5.787862	32.48437
P21	2407.152816	231.6764483	2446.133024	5.625951	45.00761
Final (P22)	2389.738702	240.8587387	2460.555523	5.642634	55.0721
Total				114.6753	4439.345

The fuel optimal trajectory was generated from the 4D waypoint network using the Dijkstra's shortest path algorithm. The fuel optimal trajectory contains 9 waypoints [initial (P1)→ P2→ P3→ P4→ P5→ P18→ P19→ P11→ final (P22)]. The comparison of fuel consumed in different phases of flight for these two trajectories and fuel optimal trajectory are shown in (table 3).

Table 3: Fuel consumed from initial to the final waypoint in different trajectories for short-haul flight.

Trajectory	Fuel consumed [kg]			Total [kg]
	Climb	Cruise	Descent	
1	1469.1	2757.9	136.9	4363.9
2	1573.1	2724.5	141.7	4439.3
Fuel optimal	1469.1	2652.8	136.1	4258

As seen in (table 3), by using the fuel optimal trajectory for the short-haul flight (Lisbon – Geneva) the aircraft consumes 105.9 kg of less fuel than the first trajectory, which is equivalent to 2.4% less fuel than the first trajectory and consumes 181.3 kg of less fuel than the second trajectory, which is equivalent to 4.1% less fuel than the second trajectory. The fuel optimal trajectory in 3D is shown in (figure 3) where, the fuel optimal trajectory is represented by the blue line and the red circles around the trajectory are the waypoints.

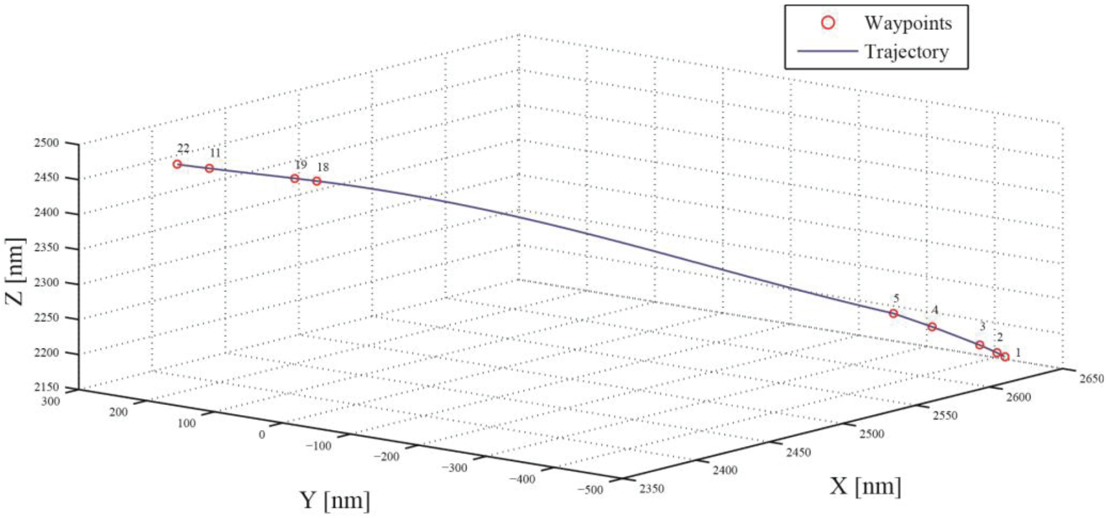

Figure 3: 3D fuel optimal trajectory in geocentric coordinates for short-haul flight

Example 2

In this example a medium-haul flight, Lisbon to Stockholm was considered. There are also two trajectories between the initial and final waypoints in the 4D waypoint network, each trajectory has 13 waypoints including the initial and final waypoints, and total 24 waypoints are there in the 4D waypoint network including the initial and final waypoint. Boeing 777-200 (B772) aircraft was used to analyze the flight trajectories. (table 4 and 5) Show the waypoints lists for both of the trajectories.

TABLE 4: List of waypoints in 1st trajectory for medium-haul flight

waypoint	$x[m]$	$y[m]$	$z[m]$	$d\tau_k$ [min]	df_k [kg]
Initial (P1)	2647.235288	-421.1992558	2155.785264	0	0
P2	2643.010993	-415.6490463	2163.819992	2.665515	1070.631
P3	2628.576005	-399.552819	2187.815353	5.529098	1724.94
P4	2599.967218	-365.4324126	2230.633603	8.291856	1859.863
P5	2586.230618	-353.564446	2249.330174	3.243938	589.0991
P6	2385.564946	-198.309008	2477.014631	42.43539	5385.051
P7	2238.11589	42.12084785	2617.270998	39.21055	4975.819
P8	2030.506669	324.3682322	2761.81912	47.18143	5987.324
P9	1782.896372	459.677681	2908.641706	39.59531	5024.645
P10	1772.18145	466.7142505	2913.251798	1.695762	31.54117
P11	1731.625729	489.6688411	2931.15743	6.807662	161.8522
P12	1690.948931	517.3407314	2947.282952	9.043191	287.5735
Final (P24)	1676.867259	536.4414256	2950.540034	5.538229	212.0034
Total				211.2379	27310.34

Table 5: List of waypoints in 2nd trajectory for medium-haul flight

waypoint	$x[m]$	$y[m]$	$z[m]$	$d\tau_k$ [min]	df_k [kg]
Initial (P1)	2647.235288	-421.1992558	2155.785264	0	0
P13	2645.498539	-411.4598298	2161.594618	2.874146	1154.43
P14	2632.238338	-391.2773722	2184.925324	5.748028	1793.241
P15	2604.511324	-357.1441614	2226.697876	8.141368	1826.109
P16	2592.536805	-337.3252479	2244.591415	3.642769	661.5269
P17	2454.226514	-11.8794812	2417.526975	49.00066	6218.184
P18	2214.11542	178.8695328	2631.773393	46.56667	5909.311
P19	1954.552046	377.266035	2809.170706	46.27695	5872.545
P20	1805.357884	485.112201	2890.720914	25.06379	3180.595
P21	1792.670589	493.4471453	2896.374754	2.016462	37.50619
P22	1747.783059	517.5842066	2916.832487	7.488843	178.0473
P23	1699.492739	535.9784473	2939.082013	9.827318	312.5087
Final (P24)	1676.867259	536.4414256	2950.540034	5.864875	224.5074
Total				212.5119	27368.51

The fuel optimal trajectory contains 9 waypoints [initial waypoint (P1)→ P2→ P3→ P4→ P5→ P20→ P21→ P23→ final waypoint (P24)], which was generated implying Dijkstra's algorithm. The comparison of consumed fuel in different phases of flight for different trajectories including the fuel optimal trajectory are shown in (table 6).

From the initial waypoint to reach the final waypoint using the fuel optimal trajectory the aircraft consumes 579.2 kg of less fuel than the first trajectory and consumes 637.4 kg of less fuel than the second trajectory. In another word by using the fuel optimal trajectory for the medium-haul flight, the aircraft consumes 2.1% less fuel than the first trajectory, and 2.3% less fuel than the second trajectory. The fuel optimal trajectory in 3D is shown in (figure 4).

Table 6: Fuel consumed from initial to the final waypoint in different trajectories for medium-haul flight

Trajectory	Fuel consumed [kg]			Total [kg]
	Climb	Cruise	Descent	
1	5244.5	21372.8	692.97	27310.3
2	5435.3	21180.6	752.6	27368.5
Fuel optimal	5244.5	20744.4	742.2	26731.1

The blue curve in (Figure 4) corresponds to the fuel optimal trajectory for the medium-haul flight and the red circles around the curve are the waypoints of the fuel optimal trajectory.

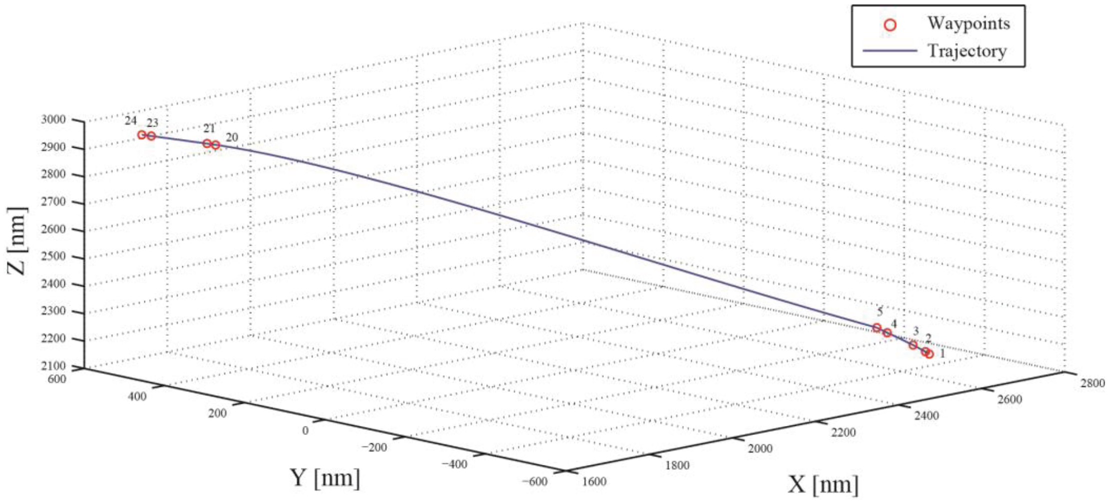

Figure 4: 3D fuel optimal trajectory in geocentric coordinates for medium-haul flight

7. Conclusion

This study is based on finding the fuel optimal trajectories of the climb, cruise and descent phases of the flight, but ignores the takeoff and landing phases of the flight. In this work, several steps were made in order to achieve a complete trajectory from a 4D waypoint network that optimizes the fuel consumption. This study uses Dijkstra's shortest path algorithm that finds a fuel optimal trajectory from a given 4D waypoints network, this technique was used to compare different length (short and medium-haul) flights.

The analysis results show promising potential for reduction of consumed fuel in different flights via using the Dijkstra's shortest path algorithm, across a range of common aircraft and routes. The results suggest that by flying fuel optimal trajectory for a short-haul flight, it is possible to save 2.4-4.1% on fuel burn, which is equivalent to 105.9 – 181.3 kilograms of fuel for B737 aircraft. In medium-haul flight by flying the fuel optimal trajectory can potentially save 2.1-2.3% fuel, reducing fuel burn by 579.2 – 637.4 kilograms for B772 aircraft. In general, the savings of the fuel is proportional to the trip lengths, and depends on the aircraft types.

Future work will deal with the extension of the proposed concept with computational intelligence methods such as the A* algorithm, reinforcement learning, and adaptive dynamic programming.

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