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Some Properties of Representation of Quaternion Group

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Abstract

The quaternions are a number system in the form $a + bi + cj + dk$. The quaternions ± 1, $\pm i$, $\pm j$, $\pm k$ form a *non-abelian* group of order eight called quaternion group. Quaternion group can be represented as a subgroup of the general linear group $GL_2(\mathbb{C})$. In this paper, we discuss some group properties of representation of quaternion group related to Hamiltonian group, solvable group, nilpotent group, and metacyclic group.

Keywords: representation of quaternion group, hamiltonian group, solvable group, nilpotent group, metacyclic group

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1. Introduction

First we review that quaternion group, denoted by Q_8, was obtained based on the calculation of quaternions $a + bi + cj + dk$. Quaternions were first described by William Rowan Hamilton on October 1843 [1]. The quaternion group is a *non-abelian* group of order eight.

2. Materials and Methods

Here we provide Definitions of quaternion group dan matrix representation.

Definition 1.1. [2] The *quaternion group*, Q_8, is defined by

$$Q_8 = \{1, \quad -1, \quad i, \quad -i, \quad j, \quad -j, \quad k, \quad -k\} \tag{1}$$

with product computed as follows:

- $1 \cdot a = a \cdot 1 = a$, for all $a \in Q_8$
- $(-1) \cdot a = a \cdot (-1) = -a$, for all $a \in Q_8$
- $i \cdot i = j \cdot j = k \cdot k = -1$

- $i \cdot j = k$, $j \cdot i = -k$
- $j \cdot k = i$, $k \cdot j = -i$
- $k \cdot i = j$, $i \cdot k = -j$

For every $a, b \in Q_8$, $a \cdot b \neq b \cdot a$. Thus Q_8 is a *non-abelian* group.

Definition 1.2. [3] A *matrix representation* of degree n of a group G is a *homomorphism* ρ of G into general linear group GL_n(**F**) over a field **F**.

It means that for every $x \in G$ there corresponds an $n \times n$ matrix $\rho(x)$ with entries in **F**, and for all $x, y \in G$, [4]

$$\rho(xy) = \rho(x)\rho(y). \tag{2}$$

Quaternion group Q_8 can be represented by matrices, i.e. matrices of general linear group GL_2(**C**) over complex vector space **C**.

According to Marius Tarnauceanu [5], quaternion group is usually defined as a subgroup of the general linear group GL_2(**C**) consisting of 2×2 matrices with unit determinant called special linear group SL_2(**C**).

A homomorphism $\rho : Q_8 \to SL_2$(**C**) of quaternion group Q_8 into the special linear group SL_2(**C**) over a complex vector space is given by:

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad -1 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$i \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad -i \mapsto \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad -j \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$k \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad -k \mapsto \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

Since all of the matrices above have unit determinant, the homomorphism ρ is the representation of quaternion group into SL_2(**C**) under matrix mutiplication.

Suppose $Q = \{I, -I, A, -A, B, -B, C, -C\}$ be a representation of quaternion group given by:

$$\rho(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \qquad \rho(-1) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

$$\rho(i) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = A \qquad \rho(-i) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -A$$

$$\rho(j) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = B \qquad \rho(-j) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -B$$

$$\rho(k) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = C \qquad \rho(-k) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -C.$$

Here we present some Definitions related to Hamiltonian group, solvable group, nilpotent group, and metacyclic group.

A *Dedekind* group is a group in which every subgroup is normal. Every subgroup in an *abelian* group is normal, hence all *abelian* groups are Dedekind. But there also exists *non-abelian* group in which all of its subgroup are normal.

Definition 1.3. [6] A *non-abelian* Dedekind group is called a *Hamiltonian* group.

Let *H* be a normal subgroup of *G*. For any $a \in G$, the set $aH = \{ah \mid h \in H\}$is called the *left coset* of G in *H*. And the set $Ha = \{ha \mid h \in H\}$is called the *right coset* of *G* in *H*.

Let *H* be a normal subgroup of *G*, then the set of right (or left) cosets of *H* in *G* is itself a group called the *factor group* of *G by H* , denoted by G/H.

Definition 1.4. [3] A group G is called *solvable* if there exist a normal series from group *G*

$$G = N_0 \supset N_1 \supset N_2 \supset \cdots \supset N_i = \{e\} \tag{3}$$

such that each N_i is normal in N_{i-1} and the factor group N_{i-1}/N_i is *abelian.*

Definition 1.5. [7] If subgroup$H \leq G$ and subgroup$K \leq G$, then *commutator subgroup*

$$[H, K] = \{[h, k] \mid h \in H \text{ and } k \in K\} \tag{4}$$

where $[h, k]$ is the commutator $hkh^{-1}k^{-1}$.

Let the (*ascending*) *central series* of a finite group G be the sequence of subgroups

$$\{e\} = Z_0(G) \subset Z_1(G) \subset Z_2(G) \subset \cdots . \quad (5)$$

And the characteristic subgroups $Z_i(G)$ of group $G$is defined by induction:

$$Z_1(G) = G; Z_{i+1}(G) = \left[Z_i(G), G\right] \text{ for } i \geq 1. \quad (6)$$

The commutator subgroup, characteristic subgroups $Z_i(G)$, and the central series of a group G lead to the following definition.

Definition 1.6. [7] A group G is called *nilpotent* if there is an integer c such that $Z_{c+1}(G) = \{e\}$, and the least such c is called the *class* of the nilpotent group G.

Definition 1.7. [8] A group G is called *cyclic* if Gcan be generated by an element $x \in G$ such that $G = \{x^n \mid n \in \mathbb{Z}\}$, *n* is an element of integers.

Such an element x is called a *generator* of G. G is a cylic group generated by x is indicated by writting $G = \langle x \rangle$.

Definition 1.8. [9] A group G is *metacyclic* if it contains a cyclic normal subgroup N such that G/N is also cyclic.

3. Result and Discussion

Here we present the results from our studies related to Hamiltonian group, solvable group, nilpotent group, and metacyclic group. Some properties of representation of quaternion group are contained in some following Propositions.

Proposition 1. Representation of quaternion group is Hamiltonian.

Proof. Let $Q = \{I, -I, A, -A, B, -B, C, -C\}$ be a representation of quaternion group. There are six normal subgroups of representation of quaternion group, which are $N_1 = \{I\}$, $N_2 = \{I, -I\}$, $N_3 = \{I, -I, A, -A\}$, $N_4 = \{I, -I, B, -B\}$, $N_5 = \{I, -I, C, -C\}$, and $N_6 = \{I, -I, A, -, B, -B, C, -C\}$.

A Dedekind group is a group G such that every subgroup of G is normal. According to Definition 1.3, then representation of quaternion grup Q is Hamiltonian.

Every subgroup is normal in every *abelian* group. In the other hand, Q is a *non-abelian* group in which every subgroup is normal.

Proposition 2. Representation of quaternion group is solvable.

Proof. Let $Q = \{I, -I, A, -A, B, -B, C, -C\}$ be a representation of quaternion group. One of the normal series for Q is $N_6 \supset N_3 \supset N_2 \supset N_1$ in which $N_2 \Delta N_1$, $N_3 \Delta N_2$, and $N_6 \Delta N_3$. There are three factor groups in that normal series, which are N_2/N_1, N_3/N_2, and N_6/N_3.

For all matrices $X, Y \in N_2/N_1$, $XY = YX$ under matrix multiplication. Hence N_1/N_2 is *abelian*. For all matrices $X, Y \in N_3/N_2$, $XY = YX$ under matrix multiplication. Hence N_3/N_2 is *abelian*. And for all matrices $X, Y \in N_6/N_3$, $XY = YX$ under matrix multiplication. Hence N_6/N_3 is *abelian*. Hence all of the factor groups in normal series $N_6 \supset N_3 \supset N_2 \supset N_1$ of Q are *abelian*.

According to Definition 1.4, since there exists a normal series from Q such that each factor group is *abelian*, thus Q is solvable.

Proposition 3. The representation of quaternion group is nilpotent.

Proof. Let $Q = \{I, -I, A, -A, B, -B, C, -C\}$ be a representation of quaternion group. The sequence of subgroups of Q in given by $\{I\} = Z_0(Q) \subset Z_1(Q) \subset Z_2(Q) \subset \cdots$. And the characteristic subgroups $Z_i(Q)$is defined by induction $Z_1(Q) = Q$, and $Z_{i+1}(Q) = \left[Z_i(Q), Q\right]$.

Thus we have the following results:

- We have $Z_2(Q) = \left[Z_1(Q), Q\right] = [Q, Q]$. Notice that the commutator subgroup $[Q, Q]$ is the set of all commutator $[X, Y] = XYX^{-1}Y^{-1}$ where matrix $X \in Q$ and matrix $Y \in Q$.

$$
\begin{aligned}
[Q, Q] &= \{\ [X, Y]\ \mid \text{matrix X} \in Q, \text{matrix Y} \in Q\} \\
&= \{I, -I\} \\
&= N_2
\end{aligned}
$$

Hence we have $Z_2(Q) = N_2$.

- Next we have $Z_3(Q) = \left[Z_2(Q), Q\right] = \left[N_2, Q\right]$.

$$\begin{aligned}\left[N_2, Q\right] &= \left\{\ [X, Y]\ \middle|\ \text{matrix X} \in N_2, \text{matrix Y} \in Q\right\}\\ &= \{I\}\\ &= N_1\end{aligned}$$

Hence we have $Z_3(Q) = \{I\}$.

According to Definition 1.6, since there is an integer $c = 2$ such that $Z_3(Q) = Z_{2+1}(Q) = \{I\}$, thus Q is nilpotent, and the class of the nilpotent group Q is 2.

Proposition 4. Representation of quaternion group is metacyclic.

Proof. Let $Q = \{I, -I, A, -A, B, -B, C, -C\}$ be a representation of quaternion group. Based on Proposition 1, normal subgroups of representation of quaternion group Q are $N_1 = \{I\}$, $N_2 = \{I, -I\}$, $N_3 = \{I, -I, A, -A\}$, $N_4 = \{I, -I, B, -B\}$, $N_5 = \{I, -I, C, -C\}$, and $N_6 = \{I, -I, A, -A, B, -B, C, -C\}$.

Generators of normal subgroups of representation of quaternion group Q can be described as follows:

- $$\begin{aligned}(I)^n &= \left\{\ldots, (I)^0, (I)^1, (I)^2, \ldots\right\}\\ &= \{I\}\\ &= N_1\end{aligned}$$

  We have $N_1 = \langle I\rangle$, hence N_1 is cyclic.

- $$\begin{aligned}(-I)^n &= \left\{\ldots, (-I)^0, (-I)^1, (-I)^2, (-I)^3, \ldots\right\}\\ &= \{I, -I\}\\ &= N_2\end{aligned}$$

  We have $N_2 = \langle -I\rangle$, hence N_2 is cyclic.

- $$\begin{aligned}(A)^n &= \left\{\ldots, (A)^0, (A)^1, (A)^2, (A)^3, \ldots\right\}\\ &= \{I, -I, A, -A\}\\ &= N_3\end{aligned}$$

$$\begin{aligned}(-A)^n &= \left\{\ldots,(-A)^0,(-A)^1,(-A)^2,(-A)^3,\ldots\right\}\\ &= \{I,-I,A,-A\}\\ &= N_3\end{aligned}$$

We have $N_3 = \langle A\rangle$ and $N_3 = \langle -A\rangle$, hence N_3 is a cyclic group which has two generators.

•

$$\begin{aligned}(B)^n &= \left\{\ldots,(B)^0,(B)^1,(B)^2,(B)^3,\ldots\right\}\\ &= \{I,B,-I,-B\}\\ &= N_4\end{aligned}$$

$$\begin{aligned}(-B)^n &= \left\{\ldots,(-B)^0,(-B)^1,(-B)^2,(-B)^3,\ldots\right\}\\ &= \{I,-B,-I,B\}\\ &= N_4\end{aligned}$$

We have $N_4 = \langle B\rangle$and $N_4 = \langle -B\rangle$, hence N_4 is a cyclic group which has two generators.

•

$$\begin{aligned}(C)^n &= \left\{\ldots,(C)^0,(C)^1,(C)^2,(C)^3,\ldots\right\}\\ &= \{I,C,-I,-C\}\\ &= N_5\end{aligned}$$

$$\begin{aligned}(-C)^n &= \left\{\ldots,(-C)^0,(-C)^1,(-C)^2,(-C)^3,\ldots\right\}\\ &= \{I,-C,-I,C\}\\ &= N_5\end{aligned}$$

We have $N_5 = \langle C\rangle$ and $N_5 = \langle -C\rangle$, hence N_5 is a cyclic group which has two generators.

In the other hand, normal subgroup N_6 is not cyclic because $N_6 \neq \langle X\rangle$ for any matrix $X \in N_6$.

Next, the factor group Q/N of normal subgroups of representation of quaternion group Q can be described as follows:

- Factor group

$$\begin{aligned} Q/N_1 &= \{XN_1 \mid \text{matrix } X \in Q\} \\ &= \{IN_1, -IN_1, AN_1, -AN_1, BN_1, -BN_1\} \end{aligned}$$

Factor group Q/N_1 is not cyclic because $Q/N_1 \neq \langle X \rangle$ for any matrix $X \in Q/N_1$.

- Factor group

$$\begin{aligned} Q/N_2 &= \{XN_2 \mid \text{matrix } X \in Q\} \\ &= \{IN_2, AN_2, BN_2, CN_2\} \end{aligned}$$

Factor group Q/N_2 is not cyclic because $Q/N_2 \neq \langle X \rangle$ for any matrix $X \in Q/N_2$.

- Factor group

$$\begin{aligned} Q/N_3 &= \{XN_3 \mid \text{matrix } X \in Q\} \\ &= \{IN_3, BN_3\} \end{aligned}$$

We have $Q/N_3 = \langle BN_3 \rangle$, hence factor group Q/N_3 is cyclic.

- Factor group

$$\begin{aligned} Q/N_4 &= \{XN_4 \mid \text{matrix } X \in Q\} \\ &= \{IN_4, AN_4\} \end{aligned}$$

We have $Q/N_4 = \langle IN_4 \rangle$, hence factor group Q/N_4 is cyclic.

- Factor group

$$\begin{aligned} Q/N_5 &= \{XN_5 \mid \text{matrix } X \in Q\} \\ &= \{IN_5, AN_5\} \end{aligned}$$

We have $Q/N_5 = \langle IN_5 \rangle$, hence factor group Q/N_5 is cyclic.

- Factor group

$$\begin{aligned} Q/N_6 &= \{XN_6 \mid \text{matrix } X \in Q\} \\ &= \{IN_6\} \end{aligned}$$

We have $Q/N_6 = \langle IN_6 \rangle$, hence factor group Q/N_6 is cyclic.

Thus, representation of quaternion group Q contains cyclic normal subgroups N_3, N_4, and N_5 such that factor groups Q/N_3, Q/N_4, and Q/N_5 are also cyclic. According to Definition 1.9, then Q is metacyclic.

4. Conclusions

From our results, some properties of representation of quaternion group are proved to be contained in the Propositions above, which are:

- Representation of quaternion group is Hamiltonian.
- Representation of quaternion group is solvable.
- Representation of quaternion group is nilpotent.
- Representation of quaternion group is metacyclic

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