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Mathematical Model of Heat Exchange and Approximate Methods of Solution of Radiation Transfer Equation in the Melting Furnace Tank

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Abstract

Mathematical model of heat exchange in melting furnace tank is considered. Equations of impulse balance for infinitesimal volume as well and continuity equation are simplified. Dimensionless parameters for compilation of discrete analogue are introduced. The system of equations of melt movement is made, solution of which is supposed to be performed using finite-difference methods. The approximate methods of solution of radiation transfer equation for optically thin and optically thick layers are considered. For optical thin layer expressions of spectral function of the source in the assumption of isotropic radiation and axial symmetry, intensity of radiation on the boundary areas and density of the monochromatic flux for the resulting radiation are simplified. Expression for the density of the monochromatic flux in the approximation of the optically dense layer is developed.

Keywords: melting furnace, approximate methods, heat exchange by radiation, radiation transfer equation, mathematical model

1. Introduction

The mathematical model of heat exchange, considered in the article, is intended for use in calculations of heat exchange in tanks of glass and open-hearth furnaces [1–5], but can be also used for melting furnaces of other design.

When using the mathematical model, the initial data play a significant role, and special attention shall be paid to the coefficient of heat exchange. Since heat exchange in the furnace tank is provided to a greater extent by radiation, the article describes approximate methods of solving radiation transfer equation.

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Approximate methods are important as use of calculations using integro-differential equations and engineering packages of programming causes a lot of mathematical difficulties and makes calculations too lengthy. Approximate methods solve these problems without reducing the accuracy of calculations.

2. Mathematical Model of Heat Exchange in the Melting Furnace Tank

When developing the numerical scheme of the melting furnace it is expedient to carefully consider the peculiarities of melt. The first such feature is its high density. Considering the high viscosity and slow movement (practically creeping), it is possible to consider confidently a melt as incompressible liquid. Of course, its density depends on the temperature, and temperature in the tank changes, but firstly, this dependence is weak and, secondly, melt temperature in the tank varies from 1000 to 1580C, which results in relatively minor changes in density. In other words, it is possible to consider with sufficient accuracy for melt $divv = 0$ and to write down equations of movement as follows:

– in the projection on the x-axis

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+2\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]; \tag{1}$$

– in the projection on the y-axis

$$\begin{aligned}\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right) &= -\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right] \\ &\quad +2\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right];\end{aligned} \tag{2}$$

– in the projection on the z-axis

$$\begin{aligned}\rho\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right) &= -\frac{\partial p}{\partial z}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right] \\ &\quad +\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+2\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right)-\rho g.\end{aligned} \tag{3}$$

The viscosity of the melt is essentially dependent on the temperature. However, equations (1)–(3) are equations of impulse balance for infinite small volume. In drawing up this balance it is possible to consider that melt temperature, and, consequently, and its viscosity, within the limits of elementary volume are identical. Then μ can be put out from under a derivatives sign that leads to simplification of equations.

In fact, for example, in equation (1) it is possible to allocate components:

$$\mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 v}{\partial y \partial x} + \mu \frac{\partial^2 w}{\partial z \partial x} = \mu \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0$$

and similarly for equations (2) and (3). Note that this approach does not exclude calculation of the dependence μ on temperature, so the final form of the motion equations can be represented as follows (for the *x*-axis):

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left(\mu \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \mu \left(\frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \mu \left(\frac{\partial u}{\partial z} \right).$$

For other axes, the equation is written in the same way.

The equation of continuity can also be simplified. Although we shall formally consider the presence of internal sources (effluents) of the mass, in fact we cannot do this because of the uncertainty of kinetics of physical and chemical transformations and lack of a mathematical description of this kinetics. Therefore, we will use the equation of continuity in the following form

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0.$$

Here we are somewhat contradictory to the aforementioned statement about incompressibility of melt, however, firstly, the introduction of density under the signs of derivatives does not complicate the equation, and, secondly, we prepare grounds for further accounting of heat exchange. Construction of a discrete analogue shall always be performed for dimensionless equations, because in this case it is easier to estimate the approximation and stability of the numerical scheme. Therefore, introduce the corresponding dimensionless parameters. We will calculate the components of speed in the fractions of the average flow rate of melt in the duct

$$V_0 = P/(86,4\rho_0 z_2 b_2), M/C,$$

where ρ_0 = scale value of density; *P* = furnace performance; z_2 = duct height, b_2 = duct width.

The index 'o' in the heat physical parameters characterizes the value at the scale temperature T_0. As a characteristic length we will accept length L:

$$U = \frac{u}{V_0},\quad V = \frac{v}{V_0},\quad W = \frac{w}{V_0},\quad X = \frac{x}{L},\quad Y = \frac{y}{L},\quad Z = \frac{z}{L},$$

$$H = \frac{h}{L},\quad B = \frac{b}{L},\quad Z_2 = \frac{z_2}{L},\quad B_2 = \frac{b_2}{L},\quad X_1 = \frac{x_1}{L},\quad \tilde{\rho} = \frac{\rho}{\rho_0},\quad \widetilde{\mu} = \frac{\mu}{\mu_0}.$$

Here h = depth of the melt layer in the tank; b = tank width; x_1 = length of the feed end of the tank. Substitution of these ratios in the conservative form of the equation record in the projection on the x-axis (similarly for other axes) leads to the expression:

$$\frac{\partial P}{\partial X} + \frac{\partial}{\partial X}\left(\tilde{\rho}U^2 - \frac{\widetilde{\mu}}{Re}\frac{\partial U}{\partial X}\right) + \frac{\partial}{\partial Y}\left(\tilde{\rho}UV - \frac{\widetilde{\mu}}{Re}\frac{\partial U}{\partial Y}\right) + \frac{\partial}{\partial Z}\left(\tilde{\rho}UW - \frac{\widetilde{\mu}}{Re}\frac{\partial U}{\partial Z}\right) = 0,$$

where $P = p/(\rho_0 V_0^2)$ = dimensionless pressure or Euler number, $Re = \rho_0 V_0 L/\mu_0$ = Reynolds number, $Fr = V_0^2/(gL)$ = Froude number.

Introduce symbols:

$$F_1 = \tilde{\rho}U^2 - \frac{\widetilde{\mu}}{Re}\frac{\partial U}{\partial X},\quad G_1 = \tilde{\rho}UV - \frac{\widetilde{\mu}}{Re}\frac{\partial U}{\partial Y},\quad H_1 = \tilde{\rho}UW - \frac{\widetilde{\mu}}{Re}\frac{\partial U}{\partial Z},$$

$$F_2 = \tilde{\rho}UV - \frac{\widetilde{\mu}}{Re}\frac{\partial V}{\partial X},\quad G_2 = \tilde{\rho}V^2 - \frac{\widetilde{\mu}}{Re}\frac{\partial V}{\partial Y},\quad H_2 = \tilde{\rho}VW - \frac{\widetilde{\mu}}{Re}\frac{\partial V}{\partial Z},$$

$$F_3 = \tilde{\rho}UW - \frac{\widetilde{\mu}}{Re}\frac{\partial W}{\partial X},\quad G_3 = \tilde{\rho}VW - \frac{\widetilde{\mu}}{Re}\frac{\partial W}{\partial Y},\quad H_3 = \tilde{\rho}W^2 - \frac{\widetilde{\mu}}{Re}\frac{\partial W}{\partial Z}.$$

Then the equations of motion can be rewritten as:

$$\left.\begin{aligned}
&\frac{\partial P}{\partial X} + \frac{\partial F_1}{\partial X} + \frac{\partial G_1}{\partial Y} + \frac{\partial H_1}{\partial Z} = 0;\\
&\frac{\partial P}{\partial Y} + \frac{\partial F_2}{\partial X} + \frac{\partial G_2}{\partial Y} + \frac{\partial H_2}{\partial Z} = 0;\\
&\frac{\partial P}{\partial Z} + \frac{\partial F_3}{\partial X} + \frac{\partial G_3}{\partial Y} + \frac{\partial H_3}{\partial Z} + \frac{\tilde{\rho}}{Fr} = 0.
\end{aligned}\right\}$$

Solution of this system of equations is supposed to be performed using finite-difference methods.

3. Approximate Methods of Solving the Radiation Transfer Equation

The analysis of regularity of radiation propagation in the radiant and absorbing environment, made using the Bouguer [1] law, leads to the integro-differential transfer equation of the following type

$$\frac{1}{\beta_v(s)}\frac{dI_v(s,\Omega)}{ds}+I_v(s,\Omega)=S_v(s,\Omega), \tag{4}$$

where the spectral function of the source $S_v(s,\Omega)$ is as follows

$$S_v(s,\Omega)=(1-\omega_v)I_{vb}(T)+\frac{1}{4\pi}\omega_v\int_{4\pi}p(\Omega`\Omega)I_v(s,\Omega`)d\Omega`. \tag{5}$$

Here $\omega_v = \alpha_v(s)/\beta_v(s)$ = spectral albedo; $\alpha_v(s)$ = scattering factor; $\beta_v(s)$ = expansion coefficient; I_v = spectral intensity of radiation; I_{vb}= the same according to Planck for a.b.b.; $p(\Omega`\Omega)$ = probability of radiation intensity in this direction; $\Omega`\Omega = \cos\theta_0$, θ_0 = Angle between incident and scattered rays. The last term of sum of the right part (5) characterizes radiation indicatrix.

If at $s = s_0 I_v(s,\Omega) = I_0$, the formal solution (4)

$$I_v(s,\Omega)=I_{v0}\exp[-\beta_v(s`)ds`]+\int_{s_0}^{s}\beta_v(s`)S(s`,\Omega)\exp[-\int_{s_0}^{s}\beta_v(s``)ds``]ds`.$$

Mathematical difficulties arising when solving integro-differential equations with real indicatrix of radiation led to occurrence of a number of approximate methods in the theory of transfer of radiation. In the approximation of optically thin and optically thick layers (the latter is also called diffusion approximation, or Rosseland diffusion approximation) are used simplifications arising from the limit value of the thickness of the medium (other approximations are not considered here for space considerations).

Approximate methods are useful from the point of view that they provide with various simple ways to solve complex problems of radiation transfer.

4. Approximation of Optically Thin Layer

Approximation of the optically thin layer is based on the assumption that the optical thickness of the medium τ_0 is extremely small (i.e., $\tau_0 << 1$), in this case integro-exponential functions are $E_n(z) = \int_0^1 \eta^{n-2} e^{-z/\eta} d\eta$, where n = the order of the function, and exponential can be represented as

$$E_2(\tau) = 1 - O(\tau);\quad E_3(\tau) = \frac{1}{2} - \tau + O(\tau^2);\quad e^{-\tau} = 1 - \tau + O(\tau^2). \tag{6}$$

If substitute these expressions in formal solutions considered earlier, it is possible to receive relatively simple expressions for function of the source, intensity of radiation on boundary surfaces, density of flux of result radiation and other values.

5. Expression for the Source Function

The formal solution for spectral function of the source in the assumption of isotropic radiation and axial symmetry has the following form

$$S_\upsilon(\tau) = (1-\omega_\upsilon) I_{\upsilon b}[T(\tau)] + \frac{1}{2}\omega_\upsilon [I_\upsilon^+(0) E_2(\tau) + I_\upsilon^-(\tau_0) E_2(\tau_0 - \tau) + \int_{\tau`=0}^{\tau_0} S_\upsilon(\tau`) E_1(|\tau - \tau`|) d\tau`], \tag{7}$$

where T = absolute temperature; ν = frequency; $I_\upsilon^+(0)$and $I_\upsilon^-(\tau_0)$ = spectral intensity of radiation on the boundaries of the flat layer.

For optically thin layer (i.e., at $\tau_0 << 1$), substituting in (7) approximate expressions (6) and ignoring terms of sum, having the order of τ_0, we get

$$S_\upsilon(\tau) = (1 - \omega_\upsilon) I_{\upsilon b}[T(\tau)] + \frac{1}{2}\omega_\upsilon [I_\upsilon^+(0) + I_\upsilon^-(0)].$$

The absence of an integral approximation here indicates that there is no expansion of the radiation emitted by the media itself. The physical meaning of this phenomenon is in negligible small influence of the radiation selfabsorption due to the very small optical thickness of the medium.

6. Expression for Radiation Intensity on Boundary Surfaces

Consider equations for the intensity of radiation on the boundary surfaces of the isotropic scattering flat layer with diffuse reflective boundaries

$$I_v^+(0) = \varepsilon_{1v} I_{vb}(T_1) + 2r_{1v}^d \left[I_v^-(\tau_0)E_3(\tau_0) + \int_0^1 \int_0^{\tau_0} S_v(\tau` - \mu`)e^{-\tau`/\mu`} d\tau` d\mu` \right] \quad at \mu > 0, \tag{8,a}$$

$$I_v^-(\tau_0) = \varepsilon_{2v} I_{vb}(T_2) + 2r_{2v}^d \left[I_v^+(0)E_3(\tau_0) + \int_0^1 \int_0^{\tau_0} S_v(\tau` - \mu`)e^{-\tau`/\mu`} d\tau` d\mu` \right] \quad at \mu < 0, \tag{8,b}$$

where ε_{1v} and ε_{2v} = spectral hemisphere black degrees, r_{1v}^d and r_{2v}^d = spectral hemispherical diffuse reflective abilities of boundary surfaces.

Given the approximate ratios (6) and neglecting the terms of sum, having the order of τ_0, rewrite equations (8) as follows

$$I_v^+(0) = \varepsilon_{1v} I_{vb}(T_1) + r_{1v}^d I_v^-(\tau_0), \quad \mu > 0, \tag{9,a}$$

$$I_v^-(\tau_0) = \varepsilon_{2v} I_{vb}(T_2) + 2r_{2v}^d I_v^+(0), \quad \mu < 0. \tag{9,b}$$

Having solved this system regarding the intensity of radiation on the boundary surfaces, we get

$$I_v^+(0) = \frac{\varepsilon_{1v} I_{vb}(T_1) + r_{1v}\varepsilon_{2v} I_{vb}(T_2)}{1 - r_{1v}r_{2v}}, \quad I_v^-(\tau_0) = \frac{\varepsilon_{2v} I_{vb}(T_2) + r_{2v}\varepsilon_{1v} I_{vb}(T_1)}{1 - r_{1v}r_{2v}},$$

Where index d at r_{iv} is omitted for simplicity.

7. Expression for the Density of Monochromatic Flux of the Resulting Radiation

The formal solution of the radiation transfer equation for the case of isotropic scattering gives the following expression for the density of monochromatic radiation flow

$$q_v^{pe3}(\tau) = 2\pi \left[I_v^+(0)E_3(\tau) + \int_0^\tau S_v(\tau`)E_2(\tau - \tau`)d\tau` \right]$$

$$-2\pi \left[I_v^-(\tau)E_3(\tau_0 - \tau) + \int_\tau^{\tau_0} S_v(\tau`)E_2(\tau - \tau`)d\tau` \right]. \tag{10}$$

When using equation (6), expression (10) is simplified and takes the form of

$$q_v^{pe3}(\tau) = 2\pi \left\{ I_v^+(0)\left(\frac{1}{2}-\tau\right) + \int_0^{\tau} S_v(\tau`)d\tau` - I_v^-(\tau_0)\left[\frac{1}{2}-(\tau_0-\tau)\right] - \int_{\tau}^{\tau_0} S_v(\tau`)d\tau` \right\}. \tag{11}$$

Here terms of sum of x_0 remained and therefore this expression has the same order of accuracy. If neglect terms of sum τ_0, then equation (11) will be simplified and take the form of

$$q_v^{pe3} = \pi\left[I_v^+(0) - I_v^-(\tau_0)\right]. \tag{11,a}$$

If the boundary surfaces are opaque and $r_{1v} = 1-\varepsilon_{1v},\ \ r_{2v} = 1-\varepsilon_{2v}$, then, substituting expressions (9) in (11a), we get

$$q_v^{pe3} = \frac{\pi[I_{vb}(T_1) - I_{vb}(T_2)]}{\dfrac{1}{\varepsilon_{1v}} + \dfrac{1}{\varepsilon_{2v}} - 1}, \tag{11,b}$$

that is, usual expression used to calculate density of the monochromatic flux of the resulting radiation between two opaque plates separated by transparent medium.

8. Approximation of Optically Thick Layer (Approximation of Rosseland, or Diffusion Approximation)

The medium is called optically thick, if average length of photon free path (i.e., the value reverse to coefficient of expansion) is small in comparison with its characteristic size. The main advantage of this approximation is that it gives a very simple expression for the flux density of the resulting radiation. We will further give a brief derivation of the expression for monochromatic flux density in the approximation of an optically dense layer.

Write formal solutions of the radiation transfer equation for the density of monochromatic flux $q_v^{pe3}(\tau)$ and spectral function of the source $S_\nu(\tau)$:

$$\begin{aligned} q_v^{pe3}(\tau) &= 2\pi\left[\int_0^1 I_v^+(0,\mu)e^{-\tau/\mu}\mu d\mu + \int_0^{\tau} S_v(\tau`)E_2(\tau-\tau`)d\tau`\right] \\ &\quad -2\pi\left[\int_0^1 I_v^-(\tau_0,-\mu)e^{-(\tau_0-\tau)/\mu}\mu d\mu \int_{\tau}^{\tau_0} S_v(\tau`)E_2(\tau-\tau`)d\tau`\right] \end{aligned} \tag{12}$$

and

$$S_{\upsilon}(\tau) = (1-\omega_{\upsilon})I_{\upsilon b}[T(\tau)] + \frac{1}{2}\omega_{\upsilon}[\int_0^1 I_{\upsilon}^{+}(0,\mu)e^{-\tau/\mu}d\mu + \int_0^1 I_{\upsilon}^{-}(\tau_0,-\mu)e^{-(\tau_0-\tau)/\mu}d\mu + \int_{\tau`=0}^{\tau_0} S_{\upsilon}(\tau`)E_1(\left|\tau-\tau`\right|)d\tau`]. \tag{13}$$

Perform decomposition of source function $S_{\nu}(\tau)$ in Taylor's series in the vicinity τ

$$S_{\upsilon}(\tau`) = S_{\upsilon}(\tau) + (\tau` - \tau)\frac{dS_{\upsilon}(\tau)}{d\tau`}\Big|_{\tau} + \frac{1}{2!}(\tau` - \tau)^2\frac{d^2S_{\upsilon}(\tau)}{d\tau`^2}\Big|_{\tau} + ... \tag{14}$$

For optically thick medium τ, τ_0 and $(\tau_0 - \tau)$ are very large everywhere, except areas near the boundaries. Thus, the areas far from the boundaries are considered, where it can be assumed that $\tau, \tau 0$ and $(\tau_0 - \tau) >> 1$.

For great τ integro-exponential and exponential functions tend to zero

$$e^{-\tau} \to 0, \quad E_n(\tau) \to 0, \quad \tau^n E_n(\tau) \to 0 \qquad \tau \to \text{o}, \; n = 1, 2, 3, ... \tag{15}$$

Substituting expansion (14) in (12) and (13), taking in parts integrals with variable integration τ' and simplifying the received expressions with the help of equation (15), we get

$$q_{\upsilon}^{pe3}(y) = -\lambda_r\frac{dT}{dy}, \qquad \lambda_r \equiv \frac{16n^2\overline{\sigma}T^3}{3\beta_R}. \tag{16}$$

Coefficient λ_r is called the coefficient of radiant thermal conductivity by analogy with the coefficient of thermal conductivity known in the theory of thermal conductivity. Expression (24) has the same appearance as relevant expression for the heat flux density due to thermal conductivity; it can be seen that approximation of an optically thick layer describes the process of radiation transfer as a diffusion process.

The last expressions are called Rosseland approximation or diffusion approximation for density of a radiation flux, Rosseland average expansion coefficient β_R can be calculated by means of the function of radiation of the second kind.

9. Summary

On the stated equations on the basis of the method of control volume the numerical scheme which can be implemented in any programming language for automation of calculations is developed.

References

[1] Shvydkii, V. S., Spirin, N. A., Ladygichev, M. G., et al. (1999). *Elements of System Theory and Numerical Methods of Modeling of Heat and Mass Transfer Processes, Textbook for Universities*, p. 520. M.: Internet Engineering.

[2] Dzyuzer, V. Ya. (2008). Numerical modelling of heat-physical processes in glass furnaces/ Dzyuzer, V. Ya., Shvydkii, V. S., and Sobyanin, S. E. Furnace-chimney building: Thermal modes, designs, automation and ecology, in *Transactions of the III International Conference*, p. 56–76. Yekaterinburg: Engineering Thought.

[3] Dzyuzer, V. Ya. (2005). Method of processing the results of zonal calculation of external heat exchange /Dzjuzer, V. Ya., Shvydkii, V. S., and Sobyanin, S. E. Science and technology, in *Part 1: Transactions of the XXV Russian School and XXXV Ural Seminar Devoted to the 60th Anniversary of the Victory*, pp. 191–198. M.: RAS.

[4] Dzyuzer, V. Ya. (2006). Influence of the flame length on the hydrodynamics of glass furnace melting tank with U-shaped direction of flame/ Dzjuzer, V. Ya. and Shvydkii, V. S. in *Glass and Ceramics*, no. 9, pp. 5–11.

[5] Dzyuzer, V. Ya. (2006). Influence of overflow threshold on hydrodynamics of melt in glass furnace/Dzjuzer, V. Ya. and Shvydkii, V. S., in *XXVI Russian School on Problems of Science and Technologies: Brief Reports*, pp. 98–100. Yekaterinburg: UrBr RAS.